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PROPERTIES AND APPLICATIONS OF PHASE-SHIFTED RECTIFIED SINE WAVES

BY

J. TYKOCINSKI TYKOCINER

AND

LOUIS R. BLOOM

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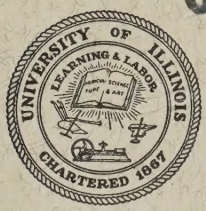

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PROPERTIES AND APPLICATIONS OF PHASE-SHIFTED RECTIFIED SINE WAVES

BY

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CONTENTS

PAGE

I. Introduction 7

1. Origin and Purpose of Investigation 7

2. Acknowledgments 7

II. Properties of Biphase Rectified Sine Waves . . . 9

3. Graphical Presentation of Properties of Wave Forms Obtained by Subtraction 9

4. Graphical Presentation of Properties of Wave Forms Obtained by Addition 10

5. Properties of Sine Waves Combined With Rectified Sine Pulses 14

III. Apparatus for Producing Phase-Shifted Rectified Sine Waves 20

6. Methods of Phase-Shifting 20

7. Phase-Shifting by Addition of Vectors in Quadrature Relation 21

8. Sound-Radiation Method of Phase-Shifting . . . 22

9. Acoustic Phase-Shifting Apparatus 23

10. Phase-Shifting by Adjusting Circuit Constants . . 26

IV. Experimental Verification of New Wave Forms and of Their Properties 27

11. Scope of Oscillographic Investigation 27

12. Experiments With Wave Forms Obtained by Vectors in Quadrature Relation at 60 Cycles 27

13. Experiments With Wave Forms Obtained by Sound-Radiation Method at 1500 Cycles 29

14. Experiments With Wave Forms Obtained by Sound-Radiation Method at 6000 Cycles 33

15. Experiments With Wave Forms Obtained by Adjusting Circuit Constants at 20 000 to 100 000 Cycles 33

PAGE

V. Mathematical Discussion of Wave Forms 36

16. General Relations 36

17. Wave Forms Obtained by Subtraction 38

18. Wave Forms Obtained by Addition 40

19. Wave Forms Obtained by Combination of Rectified Pulses With Full Sine Waves 43

20. Relations Between Wave Forms 46

VI. Applications 49

21. Review of Applications 49

22. Oscillographic Method of Phase Angle Measurement Using Difference Curve D 49

23. Oscillographic Method of Phase Angle Measurement Using Sum Curve S 50

24. Oscillographic Method of Phase Angle Measurement Using Composite Curve M 50

25. Phase Angle Measurement by Means of Direct-Reading Instruments 51

VII. Résumé of Investigation 51

26. Summary 51

27. Results 52

LIST OF FIGURES

NO.		PAGE
1.	Graphical Subtraction of Two Rectified Phase-Shifted Sine Waves	8
2.	Amplitude as Function of Phase Angle for Difference Curve D	9
3.	Graphical Addition of Two Rectified Phase-Shifted Sine Waves	11
4.	Amplitude as Function of Phase Angle for Sum Curve S	12
5.	Graphical Subtraction of a Phase-Shifted Rectified Sine Wave From a Full Sine Wave	15
6.	Amplitude as Function of Phase Angle for Difference Curve M	19
7.	Diagram of Phase-Shifting Circuit	20
8.	Component Values of Vectors for Adjusting Phase Angles	21
9.	The Acoustic Phase-Shifting Apparatus	23
10.	Diagram for Phase-Shifting by Varying Circuit Constants	25
11.	Oscillograms Showing Subtraction of Two Rectified Phase-Shifted Sine Waves	28
12.	Oscillograms Showing Addition of Two Rectified Phase-Shifted Sine Waves	30
13.	Oscillograms Showing Subtraction of a Phase-Shifted Sine Wave From a Full Sine Wave	32
14.	Oscillograms Showing Subtraction of Two Rectified Phase-Shifted Sine Waves	34
15.	Subtraction and Addition of Two Full Sine Waves	47

LIST OF TABLES

NO.		PAGE
1.	Coordinates of Critical Points and Relative Periods of Difference Curve D	10
2.	Coordinates of Critical Points and Relative Periods of Sum Curve S	13
3.	Coordinates of Critical Points and Relative Periods of Composite Curve M	18
4.	Comparison of D, S, and M Curves	48

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PROPERTIES AND APPLICATIONS OF PHASE-SHIFTED RECTIFIED SINE WAVES*

I. Introduction

1. *Origin and Purpose of Investigation.*—The development of an automatic recorder of spectral sensitivity of photoelectric surfaces† required means for so controlling a pair of modulated light sources that their relative intensities be maintained in definite phase relations. In this connection a theoretical study of rectified phase-shifted sine waves was made. Enhanced by an urgent demand for a practical method of precise determination of small phase shifts, this work has been further developed experimentally.

The purpose of this investigation is

(a) To analyze by graphical methods the properties of wave forms obtained by subtraction or addition of two phase-shifted rectified sine pulses.

(b) To show that new wave forms are also obtainable by subtraction or addition of a full sine wave and a phase-shifted rectified sine pulse.

(c) To verify experimentally the predicted properties and to illustrate them by oscillograms.

(d) To develop means of producing phase-shifted rectified sine waves whose phase angles can be precisely adjusted.

(e) To substantiate the graphical method by a mathematical discussion of the new wave forms.

(f) To devise means for the applications of the properties of the new wave forms.

2. *Acknowledgments.*—This investigation has been carried on as part of the work of the Engineering Experiment Station under the general administrative direction of Dean M. L. Enger, Director of the Engineering Experiment Station, and Professor Ellery B. Paine, Head of the Department of Electrical Engineering. Acknowledgment is due to Dr. Henry J. Miles of the Mathematics Department for checking the mathematical discussion and for making suggestions which aided a clearer presentation of Chapter V.

*A brief account of this investigation was given in a paper presented before the annual meeting of the Illinois State Academy of Science on May 8, 1942.

†J. T. Tykociner and L. R. Bloom, *Journal of the Optical Society of America;* Vol. 31, pp. 689-692; 1941.

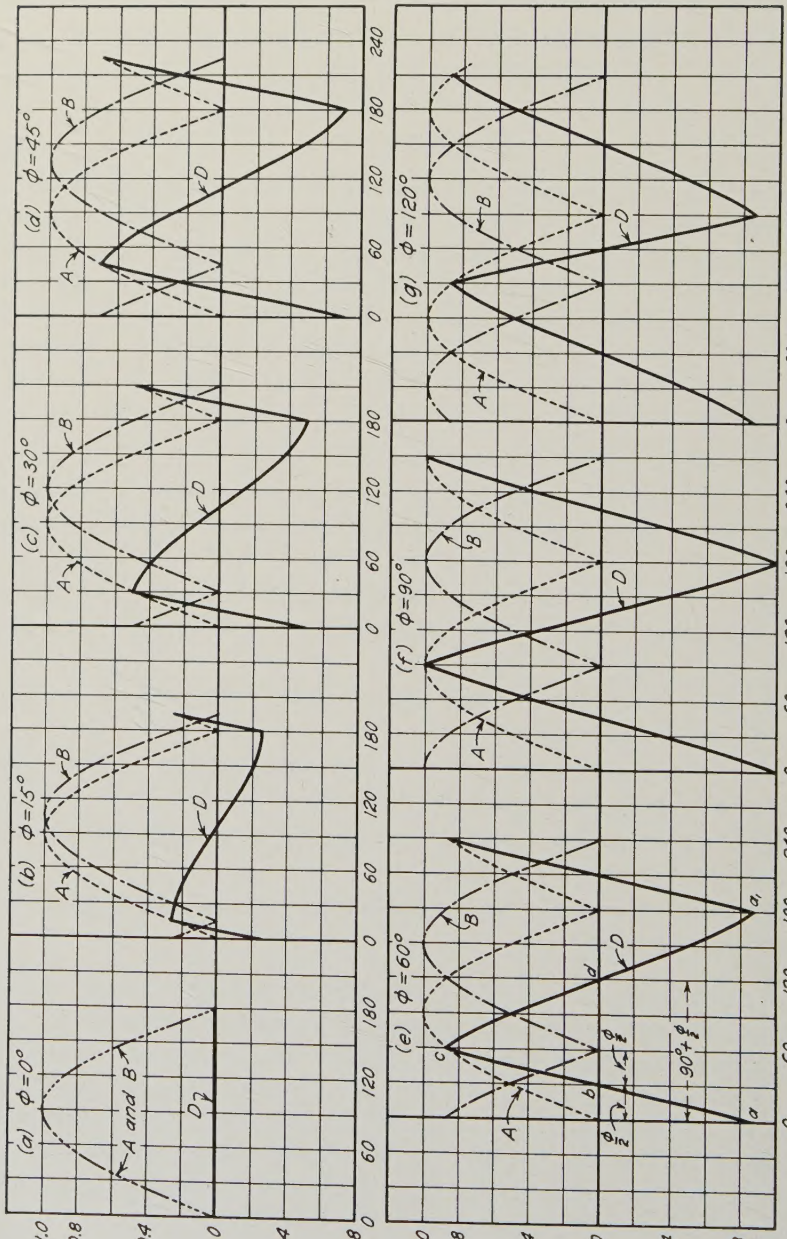


FIG. 1. GRAPHICAL SUBTRACTION OF TWO RECTIFIED PHASE-SHIFTED SINE WAVES

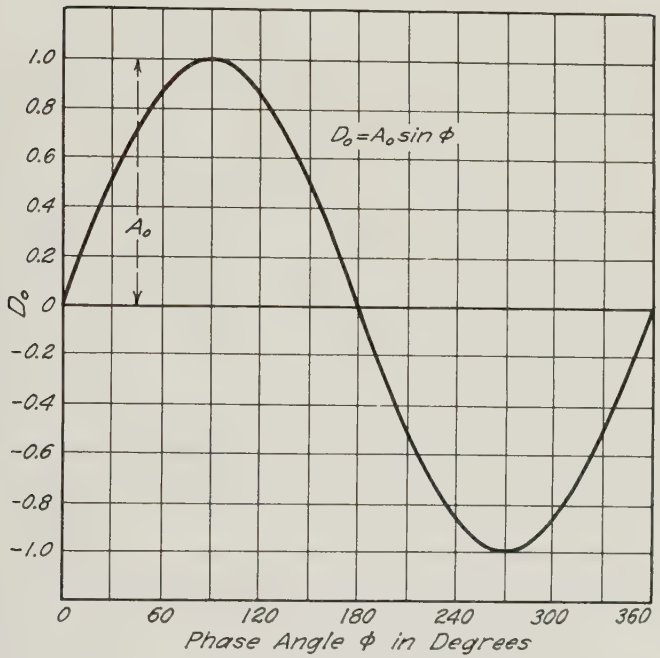


FIG. 2. AMPLITUDE AS FUNCTION OF PHASE ANGLE FOR DIFFERENCE CURVE D

II. PROPERTIES OF BIPHASE RECTIFIED SINE WAVES

3. *Graphical Presentation of Properties of Wave Forms Obtained by Subtraction.*—Figure 1 shows the results of subtraction of two rectified sine waves, A and B, of equal amplitudes $A_0 = B_0$ which are shifted in phase by 0, 15, 30, 45, 60, 90, and 120 deg. respectively. Curve D in each of the figures represents wave forms produced by the difference of the ordinates, A minus B. The critical points are marked a, b, c, d, and a_1. A study of such difference curves has revealed the following properties:

(a) The original rectified sine wave with a period $\tau_0 = 2\pi$ is transformed into alternating pulses of nearly triangular wave shapes.

(b) Each pair of superimposed phase-shifted pulses is transformed into an alternating cycle. Thereby the frequency of the resulting wave D is doubled.

(c) For phase angles corresponding to odd quadrants $\phi = 0$ to $\pi/2$ or $\phi = \pi$ to $3/2\,\pi$, etc., the part of the curve a, b, c, which faces the

TABLE 1*
COORDINATES OF CRITICAL POINTS AND RELATIVE PERIODS OF DIFFERENCE CURVE D

Critical Points	Abscissas x		Ordinates D			Periods T_D	
a	$x_a = 0$	(8a)*	$D_a = -\sin\phi$	(min)	(8b)		
b	$x_b = \phi/2$	(9a)	$D_b = 0$		(9a)		
c	$x_c = \phi$	(7a)	$D_c = \sin\phi$	(max)	(7b)	$T_D = T_0/2$	(10)
d	$x_d = \pi/2 + \phi/2$	(9b)	$D_d = 0$		(9b)		
a_1	$x_e = \pi$		$D_{a1} = \sin(\pi + \phi)$	(min)			

*Numbers in brackets refer to mathematical derivations in Chapter V.

origin and connects the minimum point a, with the maximum point c, is very close to a straight line. The other part of the curve c, d, a_1, follows more noticeably a curved line with the zero point d as its inflection point.

For phase angles corresponding to even quadrants where $\phi = \pi/2$ to π or $\phi = 3/2\,\pi$ to 2π, etc., these characteristics of the two parts of the curves are reversed.

The deviation from linearity of the c, d, a_1 part of the curve first increases with increasing angles ϕ, reaches a maximum at $\phi = \pi/6$, then decreases and becomes smallest at $\phi = \pi/2$.

At the latter phase angle ($\phi = 90$ deg.) the difference curve D is completely symmetrical and its wave form resembles an isosceles triangle.

(d) The amplitude of the difference curve is proportional to the sine of the phase differences; this is shown graphically in Fig. 2. As is evident from Fig. 2, the greatest amplitude of the difference curves occurs at a phase angle $\phi = \pi/2$.

(e) In Table I are given the equations for the abscissas and ordinates of the critical points. They show the dependence of the coordinates upon the phase angle.

The numbers in brackets refer to mathematical proofs given in Chapter V.

4. *Graphical Presentation of Properties of Wave Forms Obtained by Addition.*—If instantaneous values of two biphase rectified sine waves of equal amplitude, A_0, B_0, are added, wave forms are produced which are different from those obtained by subtraction. Figure 3 represents a series of such wave forms as they evolve by varying stepwise the phase angle ϕ from 0 to 15, 30, 45, 60, 90, and 120 deg.

*Throughout this bulletin "critical points" is to be taken to mean "points of minimum, maximum, and zero values."

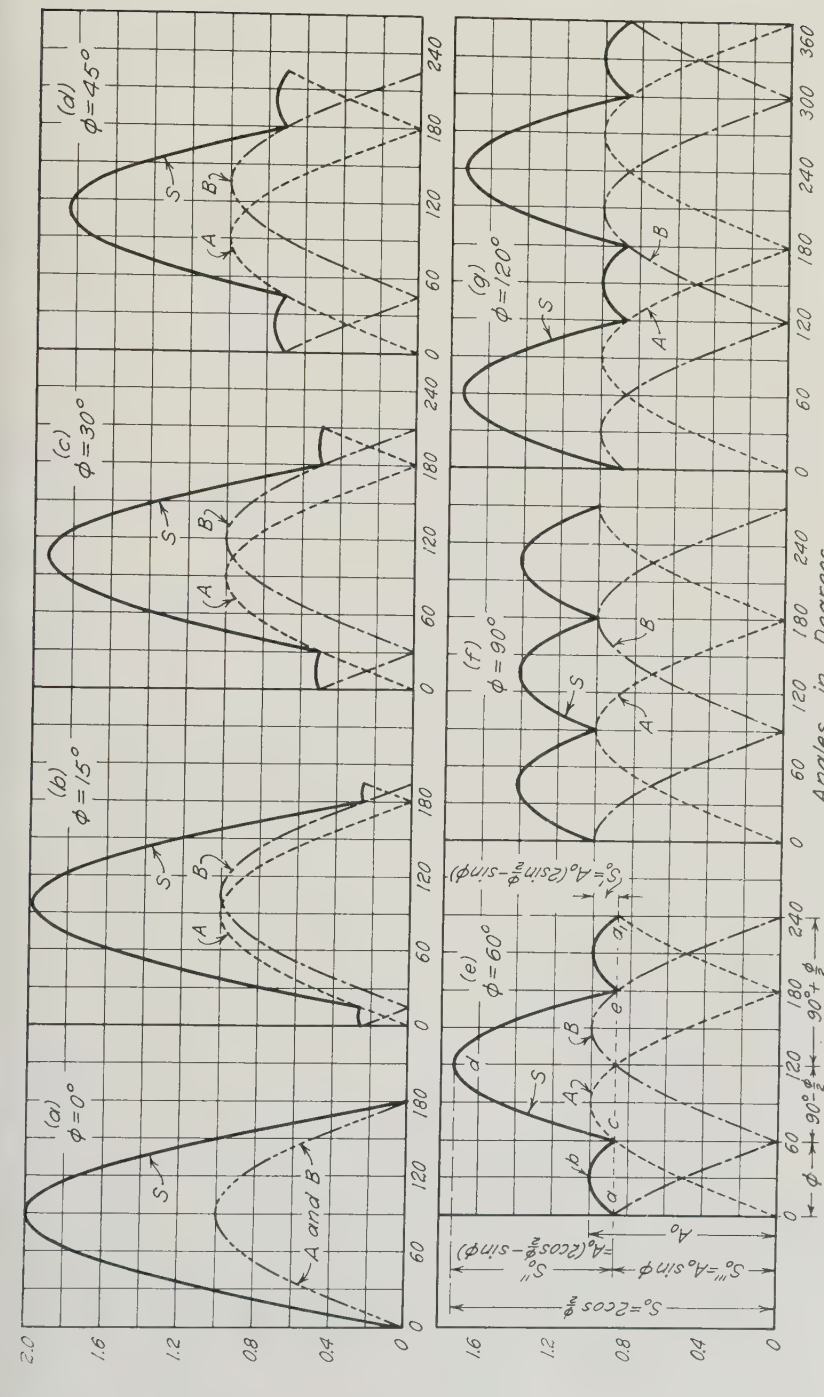


FIG. 3. GRAPHICAL ADDITION OF TWO RECTIFIED PHASE-SHIFTED SINE WAVES

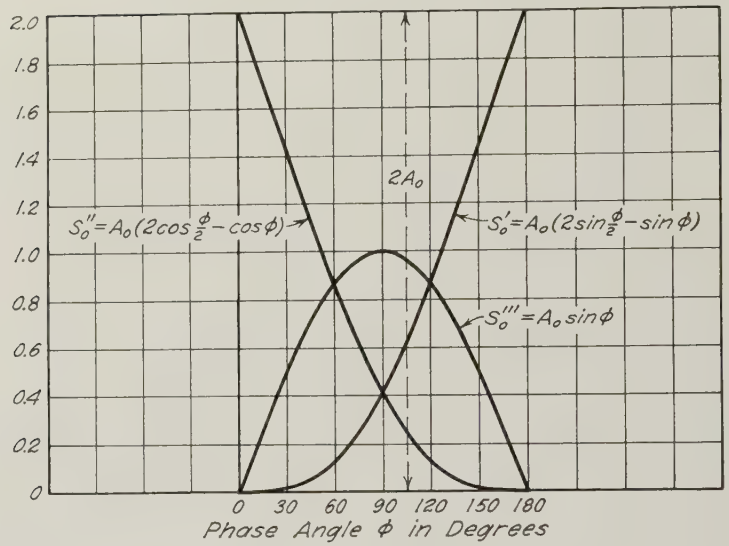


Fig. 4. Amplitude as Function of Phase Angle for Sum Curve S

The properties of these wave forms are as follows:

(a) The resultant curve S has the form of a unidirectional double pulse with conspicuous critical points a, b, c, d, and e. The ordinates may be regarded as being composed of three components.

(b) The first component a, b, c is sinusoidal in character. Its amplitude at b, denoted by S_o', depends upon the phase angle ϕ, as shown graphically in Fig. 4. This amplitude becomes zero for $\phi = 0, 2\pi$, etc., and reaches a maximum $2A_o$, or twice the amplitude of the original rectified sine pulse, for $\phi = \pi, 3\pi$, etc.

(c) The second component c, d, e is also sinusoidal in character. Its amplitude at d, denoted by S_o'', also depends upon the angle ϕ. This amplitude is a maximum, $2A_o$, at $\phi = 0, 2\pi$, etc., and reduces to zero for $\phi = \pi, 3\pi$, etc.

(d) The third component a, c, e, represented by S_o''', may be called the constant component, since it remains constant throughout the period of the pulse. Its magnitude is proportional to the phase angle ϕ, as shown in Fig. 4. This component becomes zero for $\phi = 0$, π, 2π, etc., while for $\phi = \pi/2$ it reaches a maximum value $A_o = B_o$.

(e) Tabulated in Table 2 are the equations for the abscissas X, the ordinates S, including the component amplitudes for each of the critical points as dependent upon the phase angle ϕ.

TABLE 2
COORDINATES OF CRITICAL POINTS AND RELATIVE PERIODS OF SUM CURVE S

Critical Points	Abscissas x		Ordinates S		Component Amplitudes S_0		Component Periods T_s	
a	$x_a = 0$	(17)*	$S_a = \sin \phi$	(18)				
b	$x_b = \phi/2$	(11)	$S_b = 2 \sin \phi/2$	(12)	$S_0' = 2 \sin \phi/2 - \sin \phi$	(19)	$T_s' = T_0\phi/2\pi$	(21)
c	$x_c = \phi$	(15)	$S_c = \sin \phi$	(16a)	$S_0''' = \sin \phi$	(18)	$T_s''' = \infty$	(22′)
d	$x_d = \pi/2 + \phi/2$	(13)	$S_d = 2 \cos \phi/2$	(14)	$S_0'' = 2 \cos \phi/2 - \sin \phi$	(20)	$T_s'' = T_0(\pi - \phi)/2\pi$	(22)
e	$x_e = \pi$		$S_e = \sin \phi$					

*Numbers in brackets refer to mathematical derivations in Chapter V.

(f) Figure 4 shows three characteristic intersection points P, Q, and R. Point P signifies that the component amplitudes, $S_0' = S_0'''$ at $\phi = 120$ deg., $(P = 0.866)$. Point R indicates that $S_0'' = S_0'''$ at $\phi = 60$ deg., $(R = 0.866)$. Finally, point Q marks the point at which $S_0' = S_0''$ at $\phi = 90$ deg., $(Q = 0.414)$.

(g) By assigning to the component pulses (S', S'', and S''') periods T', T'', and T''', relative to the values T_0 of the original full sine wave, relations can be expressed for each component period as enumerated in the last column of Table 2.

5. *Properties of Sine Waves Combined With Rectified Sine Pulses.*—A further series of wave forms, M, is obtained by combining sine waves with phase-shifted sine pulses of equal amplitudes $A_0 = B_0$, either by addition or subtraction of their ordinates. Figure 5 represents the wave shapes for phase angles, $\phi = 0$, 30, 60, 90, 120 deg., respectively. They are obtained by subtraction of the rectified wave B from the sine wave A, so that the resultant wave form is $M = A - B$.

The following properties of the curve M are described below:

(a) The zero axis divides the curve into a positive and negative part whose shapes and amplitudes vary depending upon the phase angle. The critical points of the curve are marked by the points a, b, c, g, d, e, f, h_1, and a_1, where a and c are the points where the curve crosses the axis. The part of the curve marked by points b, c, g, d, and e is most noticeably influenced by a change in the phase angle. Starting with $\phi = 0$ deg., when this part of the curve is a straight line coinciding with the abscissa, this line with increasing phase angle is gradually formed into a convex-concave curve with an inflection point at the point c on the abscissa axis. At angles approaching $\phi = 90$ deg., the point d marking the change in direction becomes more and more prominent until at $\phi = 90$ deg. the point e is at the terminus of two symmetrical concave curves.

(b) A similar curve M, but reversed in phase 180 deg., is obtained if the ordinates of the rectified sine wave B are added to the ordinates of the full sine wave.

(c) The abscissas X of the critical points depend on the phase angle ϕ. The corresponding functions are enumerated in Table 3.

(d) The ordinates of the critical points which differ from zero are also functions of the phase angle ϕ. They are all included in Table 3. The ordinates M_b, M_d, M_e, and M_f represent maxima and minima of the complex curve M. Their dependence on the phase angle ϕ is shown graphically in Fig. 6. The intersection point L indicates that for $\phi = 90$ deg., the amplitudes $M_d = M_f = 1.41$.

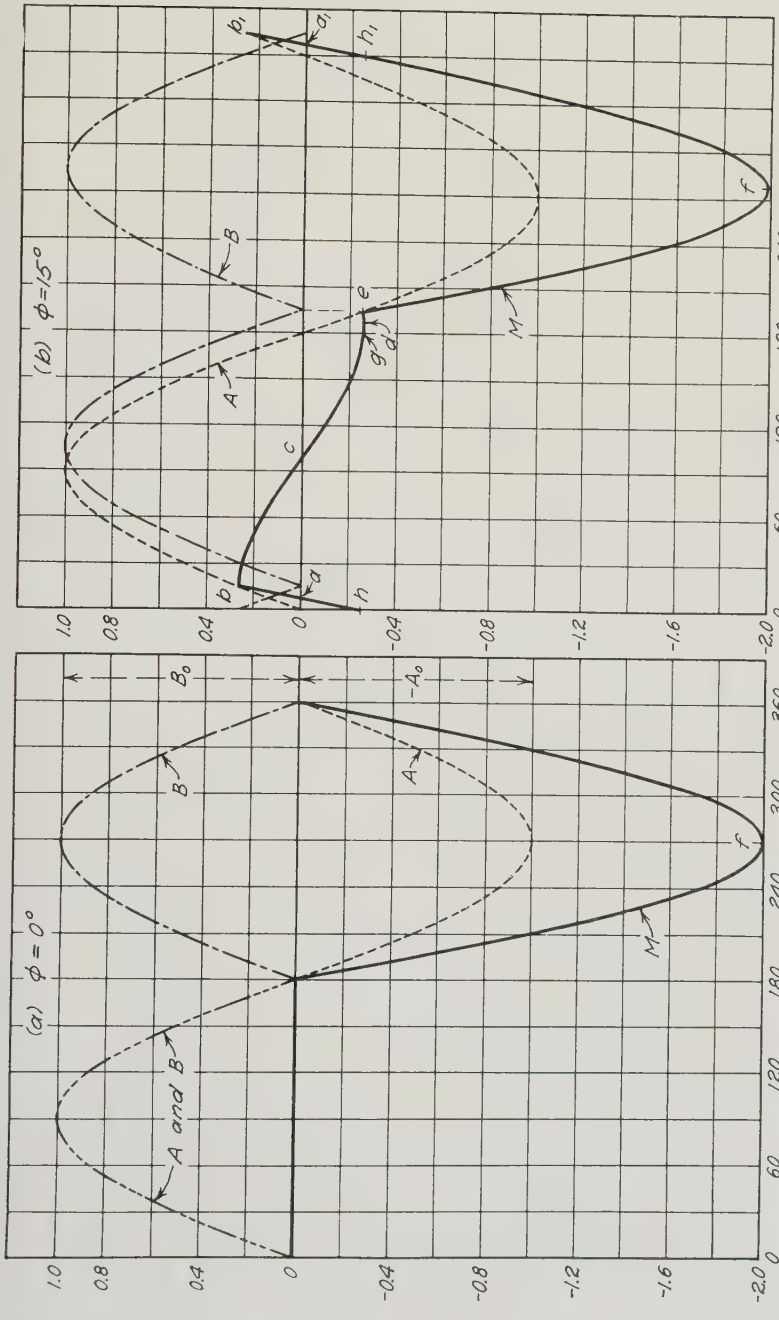


FIG. 5. GRAPHICAL SUBTRACTION OF A PHASE-SHIFTED RECTIFIED SINE WAVE FROM A FULL SINE WAVE
Phase angle $\phi = 0°$ and 15°

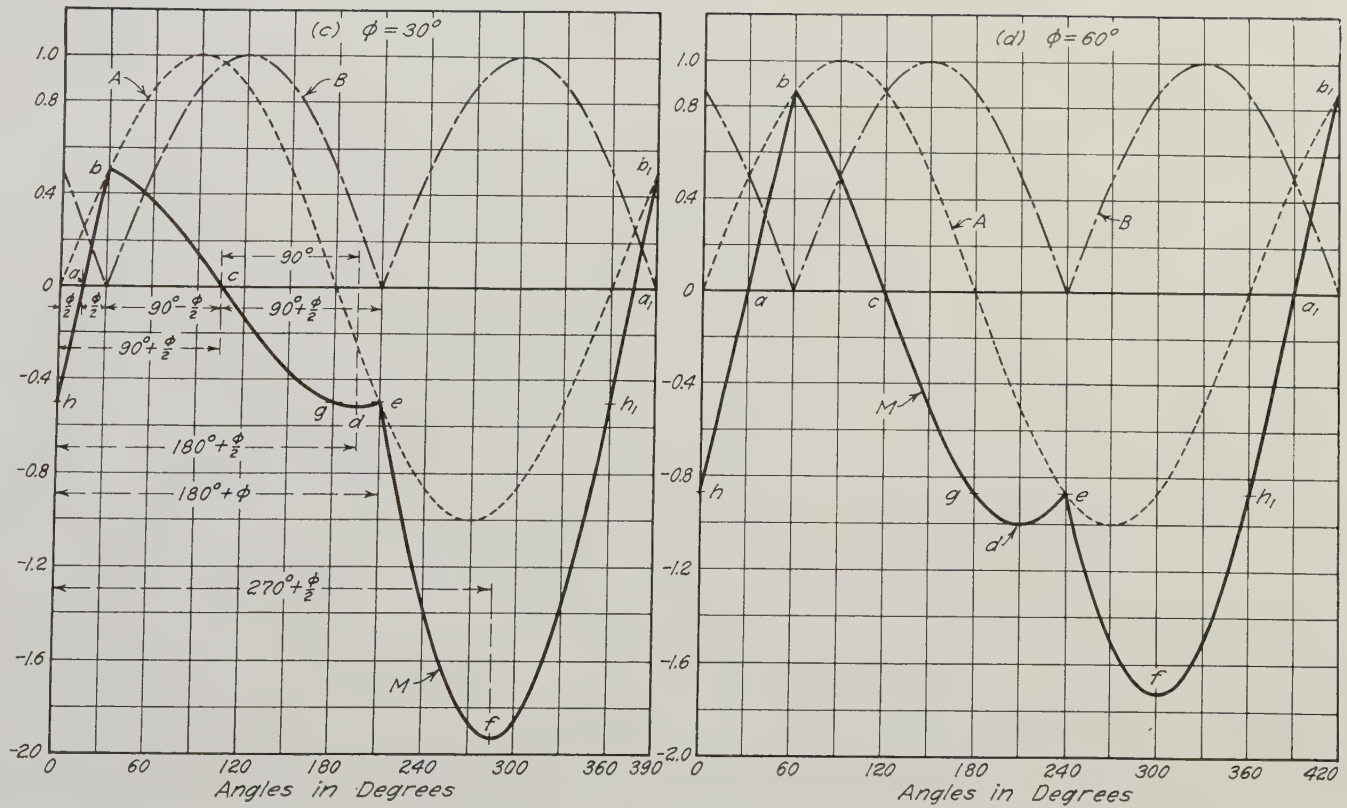


FIG. 5 (CONTINUED). GRAPHICAL SUBTRACTION OF A PHASE-SHIFTED RECTIFIED SINE WAVE FROM A FULL SINE WAVE

Phase angle $\phi = 30°$ and $60°$

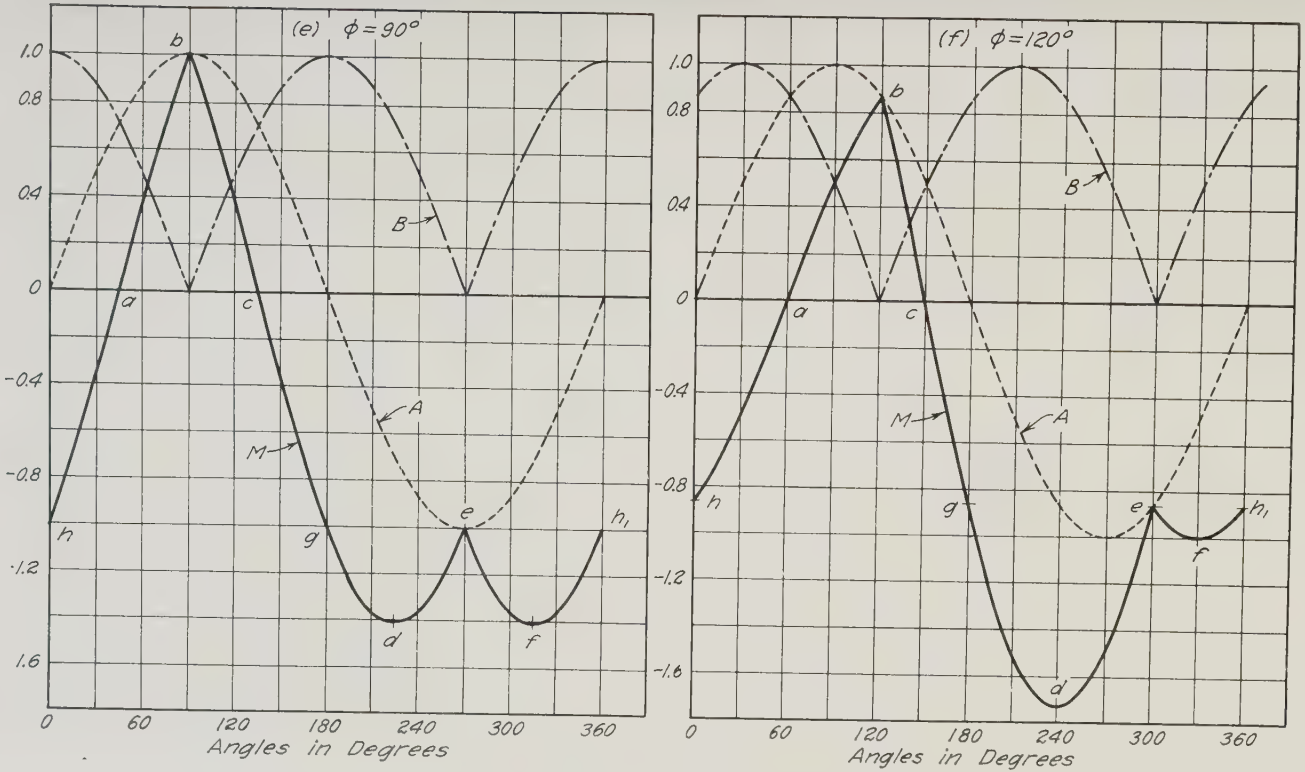


FIG. 5 (CONCLUDED). GRAPHICAL SUBTRACTION OF A PHASE-SHIFTED RECTIFIED SINE WAVE FROM A FULL SINE WAVE

Phase angle $\phi = 90°$ and $120°$

TABLE 3
COORDINATES OF CRITICAL POINTS AND RELATIVE PERIODS OF COMPOSITE CURVE M

Critical Points	Abscissas x		Ordinates M			Component Periods T_M	
a	$x_a = \phi/2$	(23)*	$M_a = 0$		(23)		
b	$x_b = \phi$	(24)	$M_b = \sin \phi$	(max)	(25)	$T_M' = T_0/4$	(33)
c	$x_c = \pi/2 + \phi/2$	(26)	$M_c = 0$		(26)		
d	$x_d = \pi + \phi/2$	(27)	$M_d = -2 \sin \phi/2$	(min)	(28)	$T_M'' = T_0(\pi + \phi)/4\pi$	(34)
e	$x_e = \pi + \phi$	(31)	$M_e = -\sin \phi$	(min)	(32)		
f	$x_f = 3/2\pi + \phi/2$	(29)	$M_f = -2 \sin (3/2\pi + \phi/2)$	(min)	(30)	$T_M''' = T_0(2\pi - \phi)/4\pi$	(35)

*Numbers in brackets refer to mathematical derivations in Chapter V.

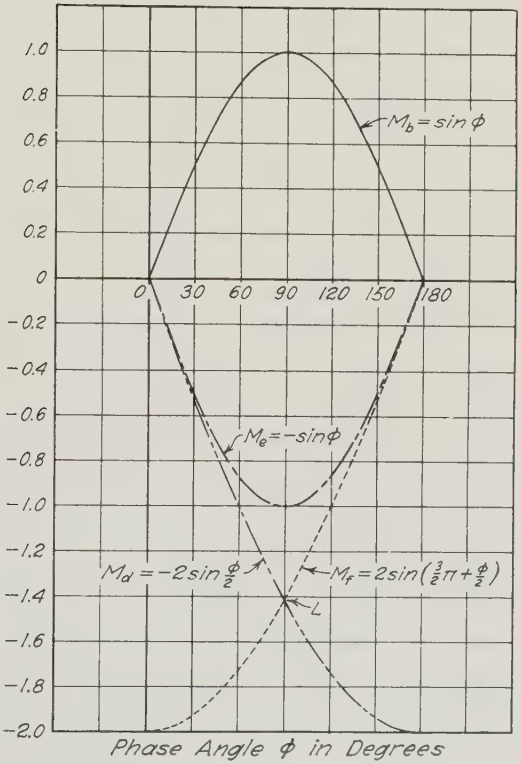


FIG. 6. AMPLITUDE AS FUNCTION OF PHASE ANGLE FOR DIFFERENCE CURVE M

(e) The complex wave form of the M curve, Fig. 5, may be regarded as composed of three pulses: the positive pulse a, b, c, the negative pulse c, g, d, e and the negative pulse e, f, h, a_1. The corresponding functions of the periods T_M', T_M'', and T_M''' which depend upon the phase angle ϕ and upon the period T_0 of the original sine wave, are enumerated in the last column of Table 3.

(f) By comparing curve M (Fig. 5) with curves D and S (Figs. 1 and 3) the resemblance of the parts of the curve M marked h, b, c, g, with the difference curve D and the part marked g, d, e, f, h, with the sum curve S becomes apparent. It is shown in Section 20 that these two component parts are identical with the D and S curves. Each of the parts follow each other at equal intervals of π radians.

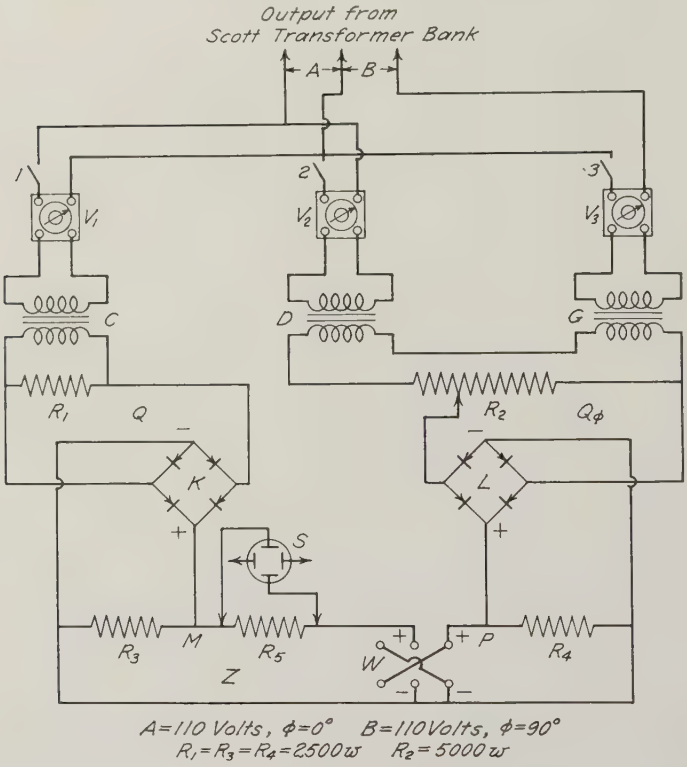


Fig. 7. Diagram of Phase-Shifting Circuit

III. Apparatus for Producing Phase-Shifted Rectified Sine Waves

6. *Methods of Phase-Shifting.*—There are many known methods for producing phase-shifted sine waves. Not all of them, however, are applicable for audio and high frequencies. In order to investigate the new wave forms over a wide range of frequencies various methods have been applied.

For the lower frequencies the well-known vectorial method has proven satisfactory. For audio frequencies an acoustical method of phase-shifting has been developed and applied for frequencies from 250 to 10 000 cycles per second. For higher frequencies up to 100 000 cycles per second the method of varying time constants of circuits has been adapted to this investigation.

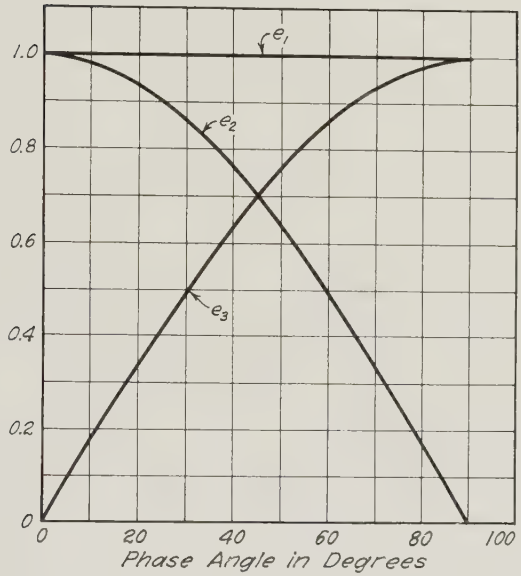


FIG. 8. COMPONENT VALUES OF VECTORS FOR ADJUSTING PHASE ANGLES

7. *Phase-Shifting by Addition of Vectors in Quadrature Relation.*— The experimental arrangement is shown in Fig. 7. It consists of a Scott-transformer bank, which supplies two 60-cycle, 110-volt sources A and B shifted 90 deg. out of phase with each other. One of the phases, A, supplies the voltage for the primaries of the transformers C and D. Variacs V_1 and V_2 are used to control their voltages. Phase B is connected to a primary of the transformer G and controlled by a third variac V_3. The secondary of the transformer C is shunted by a resistance R_1 (2500 ohms) and connected to a full-wave copper oxide rectifier K. Similarly, the secondaries of the transformers D and G are connected in series and shunted by a common resistance R_2 (5000 ohms) in the form of a potentiometer, the output actuating another full-wave rectifier L.

The d.c. outputs of the two rectifiers are connected to a mixing circuit Z whose function is to add or subtract the instantaneous values of voltage supplied by the rectifiers. This circuit consists of resistances R_3 and R_4 (2500 ohms) each shunted across one of the rectifiers. A loading resistance R_5 completes the circuit. The switch W serves to

combine the two circuits Q and Q_ϕ for addition or subtraction of the two phase-shifted rectified voltages.

The operation of the phase-shifting source is as follows: The circuit Q is adjusted for a definite amplitude e_1 as determined either on the cross-sectional screen of the oscillograph or with a voltmeter. It supplies a rectified sine pulse in phase with the supply source A. The circuit Q_ϕ supplies a voltage, which differs in phase from that of Q by an angle ϕ. The extent of the phase shift ϕ is controlled by variacs V_2 and V_3. The circuit Q_ϕ serves to add vectorially two amplitudes, e_2 and e_3, 90 deg. out of phase, whose resultant voltage is phase-shifted by an amount set by the ratios of the voltages supplied by the transformers D and G. For any required phase angle, Fig. 8 shows the relative values of the components e_1, e_2, and e_3. The rectifier L supplies a voltage whose phase ϕ relative to that of the rectifier K, can thus be arbitrarily established. Consequently, the voltage output across R_5 is the resultant of the sum or difference of the two phase-shifted rectified pulses depending upon the position of the commutating switch W.

8. *Sound-Radiation Method of Phase-Shifting.*—For frequencies of the range 250 to 10 000 cycles per second the following method of producing phase-shifted sine waves has been devised and proved successful. The principle consists in utilizing sound waves emitted from an electro-acoustic source and received by two sound-sensitive devices whose distances relative to the source are adjustable. The electromotive forces produced in the sound receivers are in phase when the respective distances to the sound source are equal. Changing the distance of one receiver changes its phase, so that the phase angle ϕ becomes

$$\phi = 2\pi \frac{d}{\lambda}$$

where d is the distance between the two receivers and λ is the wave length of the sound wave. The wave length λ is determined by the ratio

$$\lambda = \frac{v_0 + 0.607t}{f}$$

where v_0 is the velocity of sound in meters per second at a temperature of zero degree centigrade, t the room temperature, and f the frequency of the oscillator.

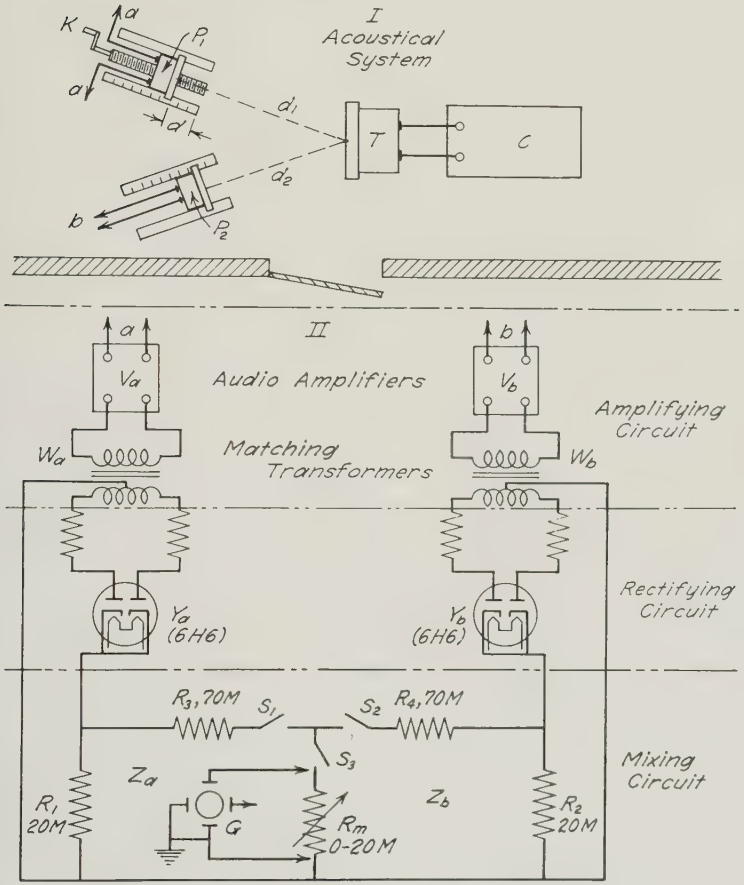


Fig. 9. The Acoustic Phase-Shifting Apparatus

9. *Acoustic Phase-Shifting Apparatus.*—A practical arrangement of an acoustic phase-shifting device constructed in accordance with the above principle is shown in Fig. 9.

In order to avoid the complications of putting the acoustic part of the phase shifter in a sound-proofed box it is convenient to separate the acoustic part I from the rest of the equipment by placing the measuring equipment II in a separate room.

Part I consists of an oscillator O whose frequency is variable from 20 cycles to 20 000 cycles per second. The oscillator is coupled to an electro-acoustic radiating source T. For frequencies not exceeding

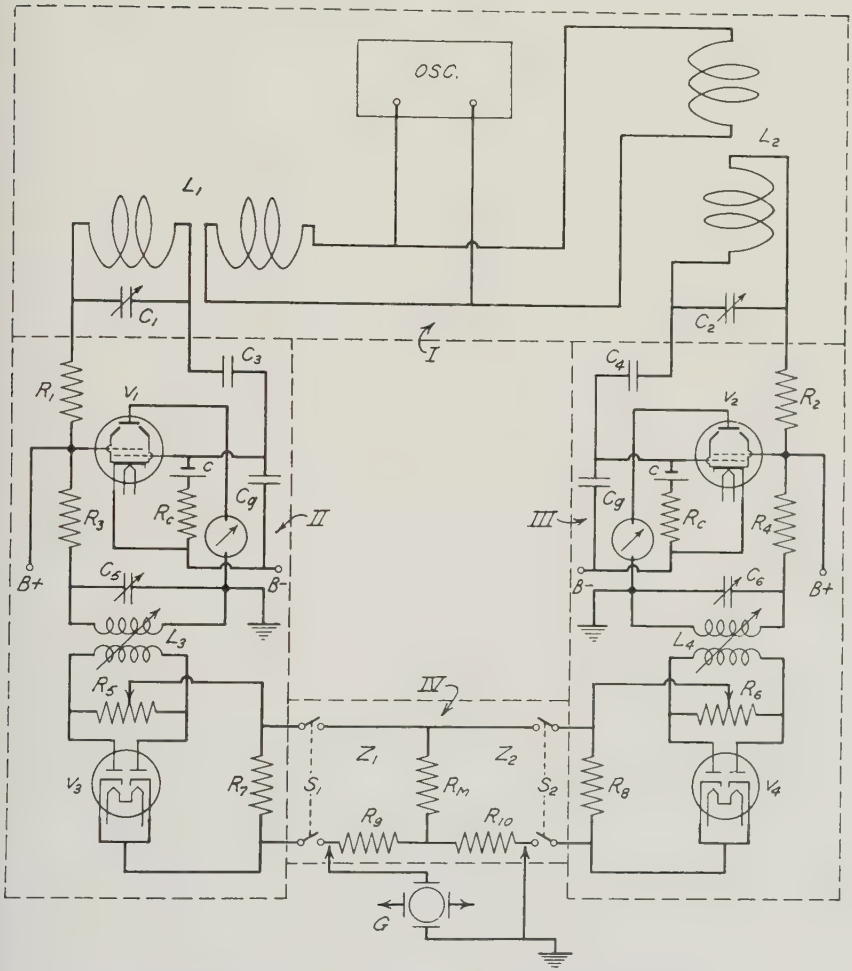
2000 cycles this transmitter T consists of a high impedance earphone. For higher frequencies this transmitter is replaced by a high fidelity dynamic speaker driving unit whose audio spectrum extends from 30 to 10 000 cycles. At distances d_1 and d_2 of approximately two wave lengths are placed two high impedance headphones P_1 and P_2, identical in their characteristics. One of the headphones, P_2, is fixed in position. The other, P_1, is mounted on a stand and can be shifted between guides by means of a worm gear. By turning a crank K, the distance d_1 can be varied and set to any desired position with an accuracy of at least ± 0.2 mm. For each given phase angle ϕ, the distance is thus determined by the relation, $d = d_1 - d_2 = \phi v/2\pi f$.

Part II consists of two sets of similar measuring equipment. Each set is made up of a voltage amplifier (V_a and V_b), a matching transformer (W_a and W_b), a full-wave rectifier (Y_a and Y_b) and a coupling circuit (Z_a and Z_b). In order to obtain precise reproduction of the wave form with little phase and amplitude distortion, high gain amplifiers of the inverse feedback type are preferable. Since the voltages are of the order of one to two volts, full wave rectifiers (6H6) designed for small voltage inputs, are used.

The outputs, a and b, of the receiver headphones P_1 and P_2 are fed through shielded concentric cables into the respective amplifiers V_a and V_b. An oscillograph G is used as an indicating instrument to observe the resultant wave forms.

With this equipment the cathode ray of the oscillograph tube is actuated by two sets of rectified sine pulses equal in amplitude and frequency but differing in phase. This phase is varied by adjusting the distance d. The phase-shifted a.c. voltages supplied by the outputs a and b, each actuating its own amplifier, V_a and V_b, are transformed into rectified sine pulses in the rectifier tubes, Y_a and Y_b. The rectified output voltages across the loading resistors R_1 and R_2 respectively, are then applied to both parts of the mixing circuits Z_a and Z_b with its component resistors R_3, R_4, and R_m.

The following combinations of wave forms are obtainable by manipulating switches S_1, S_2, and S_3, in proper combination. With all switches open, and the oscillograph plates connected across R_1 or R_2 the wave form of each of the rectified sine pulses can be traced and the amplitude adjusted to equal magnitudes by means of the gain controls of the respective amplifiers V_a and V_b. With all switches closed and the oscillograph plates connected across R_3 and R_4 the difference wave form is traced. Finally, with all switches closed but with oscillograph plates connected across R_m the resultant sum wave is traced.



OSC – Resistance-tuned Oscillator, Range 20–200 000 cps.
L_1, L_2 – 32 mh. Primary, 16 mh. Secondary.
L_3, L_4 – 10 mh. Primary, 10 mh. Secondary.
C_1, C_2 – 500 μμf., Tuning Condensers.
C_3, C_4 – 0.0001 μf., Mica.
C_5, C_6 – 1000 μμf., Tuning Condensers.
C_g – 0.01 μf., Mica.
R_1, R_2 – 25 000 Ohms, ½ Watt.
R_3, R_4 – 500 Ohms, 10 Watt.
R_5, R_6 – 20 000 Ohms, Variable Potentiometer.
R_7, R_8 – 10 000 Ohms, 1 Watt.
R_9, R_{10} – 10 000 Ohms, 1 Watt.
R_M – 4000 Ohms, 1 Watt.
R_g – 0.1 Megohm, ½ Watt.

V_1, V_2 – 6L6
V_3, V_4 – 6H6
G – Oscilloscope

FIG. 10. DIAGRAM FOR PHASE-SHIFTING BY VARYING CIRCUIT CONSTANTS

10. *Phase-Shifting by Adjusting Circuit Constants.*—The method of phase-shifting applied for radio frequencies was based on the principle of coupling an oscillator with two parallel circuits containing reactive elements whose time constants could be varied. Thus a lag between the currents was produced in the two input branches of the rectifiers. The usual precautions for operation at high frequencies had to be considered.

(1) In order to prevent the load from one of the branches of the circuit from reflecting back into the oscillator, thereby affecting the other branch, the coupling coefficient was kept as small as possible.

(2) To prevent mutual interference due to induction and radiation the two branches were separated from each other, and the shields of the leads were grounded.

(3) Distortion of the rectifier output due to stray capacitances between the leads of the rectifier tubes was avoided by shortening the leads and eliminating sockets.

Figure 10 shows the circuit diagram of the phase-shifting equipment. It consisted of four parts: an oscillator circuit, I; the two phase-shifted branches, II and III; and the mixing circuit, IV. The oscillator was coupled to the branches II and III through air core coupling transformers, L_1 and L_2, whose secondaries formed resonant circuits with condensers, C_1 and C_2.

The two transformers were placed at right angles to each other so that their fields would not affect one another. The secondary of each transformer was coupled to the input of one of the beam power amplifier tubes, V_1 and V_2 (6L6), through resistors, R_1 and R_2, and coupling condensers, C_3 and C_4, whose functions it was to prevent self-oscillation in the circuit. The output of each beam amplifier tube fed its power through the plate-biasing resistors, R_3 and R_4, into the tank circuit consisting of tuning condensers, C_5 and C_6, and the primaries of the variable air transformers, L_3 and L_4. The secondaries of the transformers were connected into the potentiometers, R_5 and R_6, which supplied the alternating voltages to the full-wave rectifier tubes, V_3 and V_4 (6H6). The voltage drop across the load resistors, R_7 and R_8, supplied the phase-shifted rectified pulses to the mixing bridge IV. The bridge consisted of double-pole single throw switches, S_1 and S_2, and the resistances, R_9, R_{10}, and R_M. An oscillograph, G, could be connected across either the resistances, R_9 and R_{10}, or the mutual resistance, R_M.

Phase-shifting was obtained by varying the condensers, C_5 and C_6, or by inserting inductance or resistance into the tuned plate circuit.

Since the amplitude is maximum at resonance when the impedance in the circuit is equivalent to the ohmic resistance, detuning by means of the condensers C_5 or C_6 would increase the impedance, and therefore decrease the amplitude by an amount which is a function of the phase angle, ϕ. For each given phase angle the ratio of these amplitudes could be calculated and thus the condensers C_5 and C_6 could be set for a value which would give the required ratio. With the phase angle established the equality of amplitudes in the output of the two branches II and III could again be obtained by adjusting the coupling of the coils of the two transformers L_1 or L_2.

IV. Experimental Verification of New Wave Forms and of Their Properties

11. *Scope of Oscillographic Investigation.*—The wave forms which result from addition or subtraction of two phase-shifted rectified sine waves are independent of frequency. However, the technique of producing and observing phase shifts depends considerably on the frequency band being studied. The description includes, therefore, the techniques developed for the various frequency ranges.

The cathode ray oscillograph offers the most direct method for tracing complex wave forms and has been chosen as the measuring instrument for the experimental verification of the new wave forms. The frequency limits are naturally set by the range for which the oscillograph can trace wave forms without distortion. With the particular oscillograph available, the range was from 30 to 100 000 cycles per second. Each method for producing phase-shifted rectified sine waves has its frequency limitations. The method of vector addition in quadrature relation has been investigated at 60 cycles per second. The acoustic method was tried for 250 to 10 000 cycles. Finally the method of adjusting circuit constants was investigated for the range of 20 000 to 100 000 cycles. Thus the application over the entire frequency spectrum covered by the oscillograph was verified. The experiments described in the following paragraphs are typical examples of the different methods for producing and observing phase differences at various frequencies.

12. *Experiments With Wave Forms Obtained by Vectors in Quadrature Relation at 60 Cycles.*—In order to verify experimentally at 60 cycles per second the wave forms of the difference curves shown graphically in Fig. 1 the method of phase-shifting by addition of vectors in quadrature relations as described in Section 7, was used.

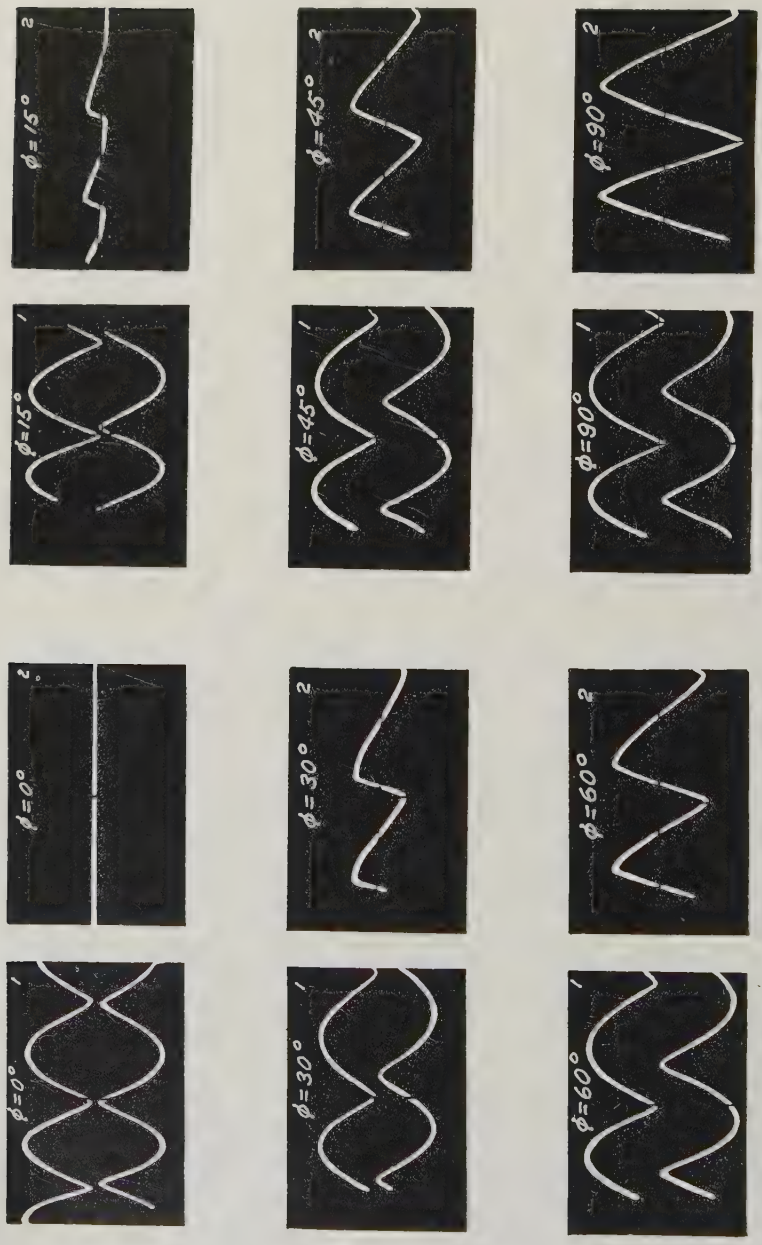


FIG. 11. OSCILLOGRAMS SHOWING SUBTRACTION OF TWO RECTIFIED PHASE-SHIFTED SINE WAVES

Figure 8 served to facilitate the evaluation of the component voltages e_1, e_2, e_3, necessary for making adjustments on the three variacs V_1, V_2, V_3, for any given phase angle ϕ. The curve was drawn on the basis of the known relations for the ratio

$$\frac{e_3}{e_2} = \tan \phi$$

and for the resultant voltage vectors

$$e_1^2 = e_2^2 + e_3^2$$

$$e_3 = \sqrt{e_1^2 - e_2^2}.$$

By solving these equations for e_3, the component e_2 was obtained, namely,

$$e_2 = \sqrt{\frac{e_1^2}{1 + \tan^2 \phi}}.$$

With this method, oscillograms, Series A, were obtained for 60 cycles per second for phase angles $\phi = 0$, 15, 30, 45, 60, 75, and 90 deg. They are shown in Fig. 11. The second vertical column in the figure gives the phase angles for each pair of oscillograms. The oscillograms in the first vertical column show separately the component wave forms of the original rectified sine pulses in their phase relations. The last column shows the resultant wave forms. The close resemblance of the latter to the difference curves D obtained by the graphical method of subtraction is fully evident by comparing Fig. 11 with Fig. 1.

13. *Experiments With Wave Forms Obtained by Sound-Radiation Method at 1500 Cycles.*—For the purpose of verifying the wave forms for the audio frequency range, oscillographic studies were made with wave forms produced by subtracting and adding phase-shifted rectified pulses, and also by the addition of sine waves and phase-shifted rectified sine pulses. The acoustic method of phase-shifting shown in Fig. 9 and described in Sections 8 and 9, was used.

The following procedure was applied: The receivers P_1 and P_2 were adjusted for $\phi = 0$ deg. by setting both at equal distances from the transmitter T. In order to ensure that ϕ was actually equal to

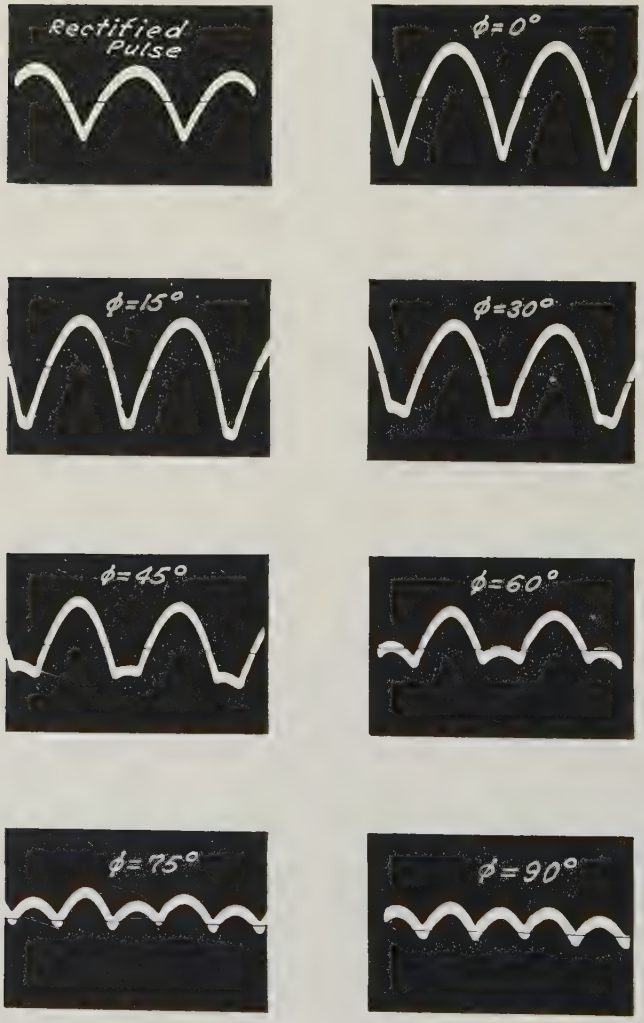


FIG. 12. OSCILLOGRAMS SHOWING ADDITION OF TWO RECTIFIED PHASE-SHIFTED SINE WAVES

zero the measuring circuits were turned on and the amplitudes, as measured with a voltmeter across R_1 and R_2, equalized. Then the final zero adjustment of the earphones was carried out and the difference wave form observed until the expected resultant wave form appeared on the screen as a straight line along the time axis in accordance with Fig. 1a. Slight phase distortions in the amplifiers were

easily corrected by a readjustment of their tone controls. When clear phase alignment was finally indicated by the straight line on the screen, P_1 was moved in the direction away from the transmitter until a point d_1 was reached corresponding to a phase shift, $\phi = 180$ deg. This was again indicated by a straight line on the screen, provided that corrections for equal amplitudes were made. Since the distance $d = d_1 - d_2$ is proportional to the phase angle ϕ, the change Δd on the scale corresponding to a 1-deg. phase shift, namely Δd per degree $= (d_1 - d_2)/180$, could be determined. This constant depends on the wave length and must be calculated for each frequency and room temperature according to the relation given in Section 8. It was equal to 0.64 mm./degree for $f = 1500$ cycles per second and $t = 25$ deg. C.

The equality of amplitudes of the outputs in the two receivers was essential for obtaining proper wave shapes. It was, therefore, found preferable to make the distance separating the receivers from the transmitter as great as possible. Thereby large corrections in the amplifier gain controls due to shifting receiver P_1 were avoided.

Examples of wave forms obtained by the sound-radiation method at a frequency of 1500 cycles per second are given by the oscillograms (Series D) shown in Fig. 12. The first oscillogram shows the wave form of one of the two rectified sine waves similar in form and equal in amplitude. All the other oscillograms show the resultant wave forms obtained by the addition of two rectified sine waves at phase differences of $\phi = 0$, 15, 30, 45, 60, 75, and 90 deg.

The oscillograms show distinctly how the wave shapes depend upon the phase angle, ϕ. With increasing phase angle the original pulse is split into two component pulses whose shapes are gradually changing into a symmetrical wave form of double the frequency at $\phi = 90$ deg. The oscillograms show also the increasing d.c. components which are determined by the distances from the minimum points of the curve to the x axis.

The relative values of the three components evaluated from the oscillograms agree with those obtained from the graphs shown in Fig. 3 and also with the mathematical expressions given in Table 2.

The same arrangement as described in Section 9, was used for oscillographic verification of the difference curves at 1500 cycles per second.

Figure 13 represents a characteristic example of oscillograms obtained with the method described in the foregoing for the case of the summation of a full sine wave and a phase-shifted rectified sine pulse.

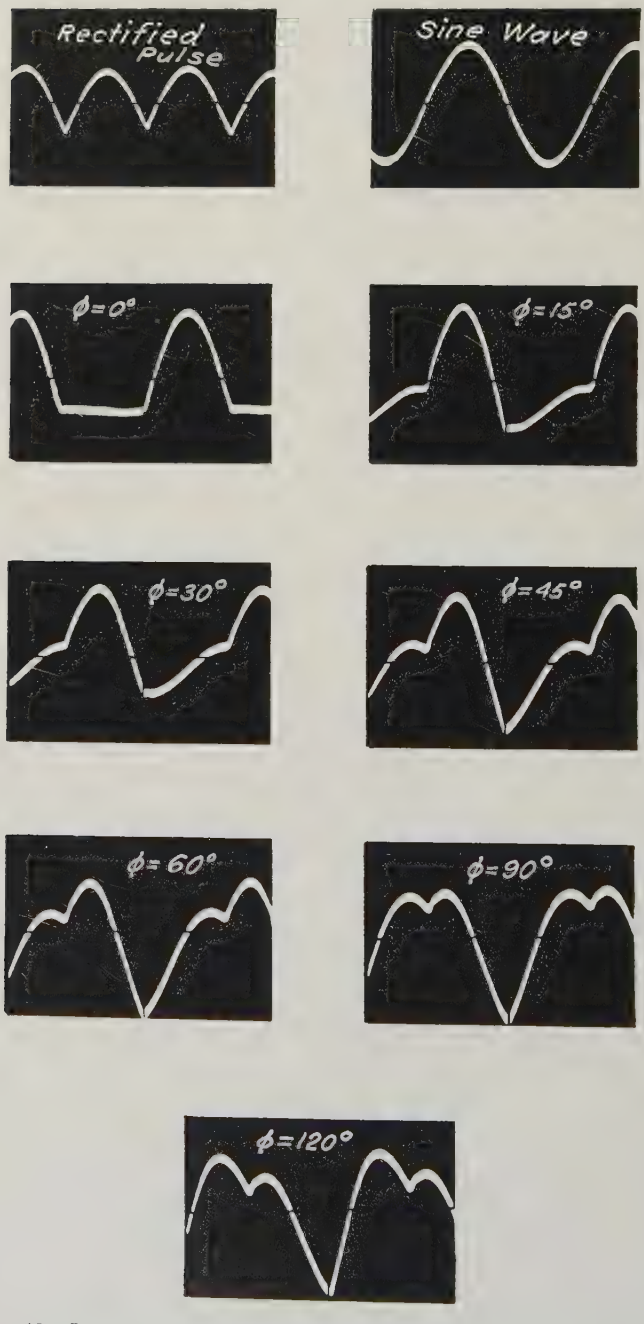


Fig. 13. Oscillograms Showing Subtraction of a Phase-Shifted Sine Wave From a Full Sine Wave

The only change as compared to Fig. 9 was that one of the rectifiers Y_b, was disconnected from the circuit and the output of the coupling transformer W_b directly connected to the load resistor, R_2. Thus, one side of the mixing circuit was excited by a full sine wave, while the other still received rectified sine pulses. With the oscillograph connected across R_M the instantaneous values of the latter curves were added and the resultant wave forms verified for $\phi = 0$, 15, 30, 45, 60, 90, and 120 deg.

A comparison of these oscillograms with Fig. 5 shows a very close agreement with wave forms obtained graphically.

The only difference to be observed is a change in phase of 180 deg. which is what was to be expected since the graph represents a subtraction while the oscillograms were obtained by summation.

All of the critical points appear as clearly and in as correct position in the oscillograms as in the graphs.

14. *Experiments With Wave Forms Obtained by Sound-Radiation Method at 6000 Cycles.*—The acoustic method as described in Section 9 (Fig. 9) had to be modified somewhat in order to obtain at the receivers P_1 and P_2 a sufficiently intense sound radiation at 6000 cycles per second. The change consisted in replacing the acoustic radiator T with a more powerful one. An annular driver unit of the permanent magnet dynamic speaker type was coupled to an oscillator feeding one watt of audio frequency energy into a 15-watt power amplifier. Although the driver was not nearly fully utilized, it, nevertheless, delivered acoustic radiation at 6000 cycles per second sufficiently intense to energize the receivers placed at a distance of four to five wave lengths from the transmitter T. The procedure of adjusting for $\phi = 0$ deg. and equalizing the amplitudes of the two component rectified pulses was exactly like the one described in the preceding section.

With this arrangement oscillograms like those shown in Series F (Fig. 14) were recorded at 6000 cycles per second. This particular series was obtained by subtraction of two phase-shifted rectified sine pulses. First is shown the wave form of one of the sine pulses. Then follow in regular sequence the difference wave forms for $\phi = 0$, 30, 45, 60, 90, and 120 deg. Finally the last oscillogram represents a repetition for $\phi = 90$ deg., but with the time axis contracted so that three times as many waves were recorded per centimeter length.

15. *Experiments With Wave Forms Obtained by Adjusting Circuit Constants at 20 000 to 100 000 Cycles.*—For the experimental verification of the wave forms obtained at frequencies between 20 000 and

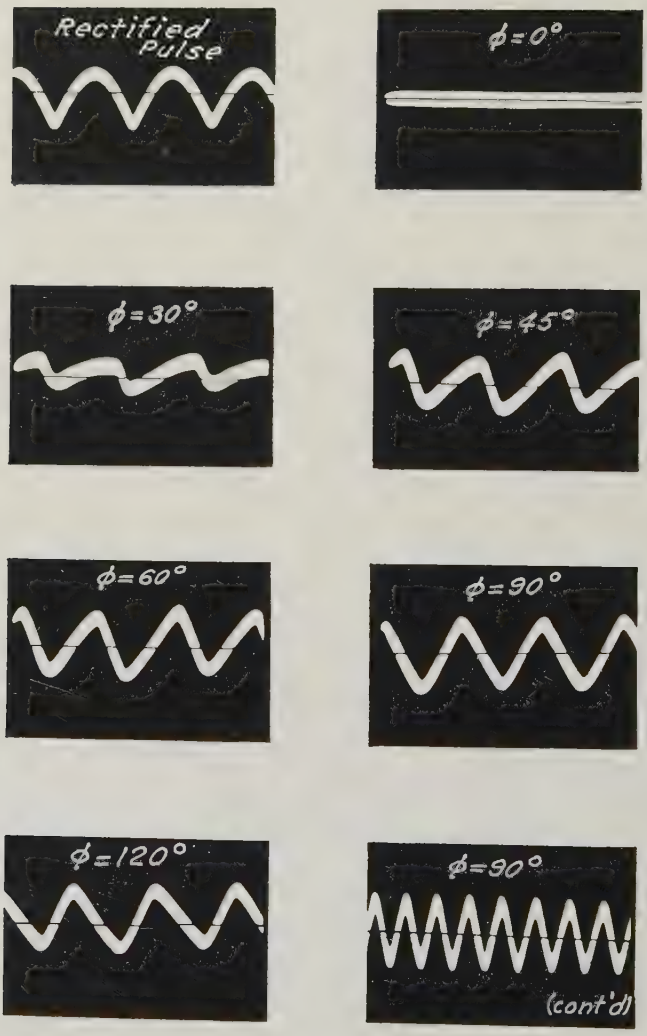


Fig. 14. Oscillograms Showing Subtraction of Two Rectified Phase-Shifted Sine Waves

100 000 cycles per second the method of phase-shifting by means of adjusting circuit constants was applied as described in Section 10 and shown in Fig. 10. The oscillator was of the resistance-tuned type whose upper frequency limit was 200 KC. The output of the oscillator (0.1 watt) was not sufficient to supply the voltage directly to the phase-shifting transformers L_3 and L_4 without distortions being pro-

duced. A tube (6L6) of the beam power amplifier type was employed to serve the following functions:

(1) To decrease third and higher order harmonics which might pass the transformers L_1 or L_2

(2) To amplify the fundamental signal supplied by the oscillator

(3) To effect coupling between the transformers with a minimum of stray fields

(4) To avoid parasitic oscillations.

The first step in the procedure consisted in adjusting the coupling of the $L_1 C_1$ and $L_2 C_2$ circuits so that loading of one circuit did not affect the amplitude of the other. Next, it was ascertained by means of the oscilloscope that no oscillations were present in the circuit other than the fundamental frequency of the oscillator, especially when the capacitances C_5 and C_6 were finally set for resonance. Further, each of the rectifier outputs was investigated separately by connecting the oscillograph first across R_7 and then across R_8 while the switches S_1 and S_2 were open. If either of the rectifiers showed that its component half waves were not equal in amplitude, the respective potentiometer arms P_1 and P_2 were shifted to balance the amplitudes. In order to equalize the amplitudes of each rectifier output, the coupling between the two coils of the transformer L_1 or L_2 was slightly changed until the two voltage amplitudes were equalized and applied to the mixing bridge.

Since the two branches were tuned to resonance ($\phi = 0$) and equalized in amplitude the image on the oscillograph screen could have only two distinct forms. The difference curve, obtained with the oscillograph connected as shown in Fig. 10, should show a straight line coinciding with the time-axis. This would verify that the resultant amplitude at $\phi = 0$ was equal to zero as expected. By connecting the oscillograph across the resistor R_M, the image should produce the sum curve which for this case ($\phi = 0$) would appear in the form of a rectified pulse whose amplitude was twice as large as that supplied by a single branch. With the completion of these preliminary tests phase-shifting of either or both of the branches could be carried out, as described in the latter part of Section 10.

The experiments were carried out at frequencies of $f = 50\ 000$, 75 000, and 100 000 cycles per second. The wave forms resembled those obtained at audio frequencies, and corresponded closely to the theoretical forms as shown in Figs. 1, 3, and 5.

At frequencies above 50 000 cycles per second slight distortions of the images were observed which consisted of a rounding of the sharp

peaks of the difference curves and the sharp edges at the base of the sum curves. It was found that these distortions were caused by stray fields across the electrodes of the rectifiers due to long input leads, binding posts, and sockets. By eliminating a great deal of these external capacitances, and also by reducing the sizes of the switches, it was possible to obtain clear images at frequencies of 80 000 to 100 000 cycles per second.

For frequencies exceeding 100 000 cycles per second both an oscillograph and rectifier would have to be designed with the object in view of reducing stray capacitances and inductances of the amplifiers.

V. Mathematical Discussion of Wave Forms

16. *General Relations.*—Most of the properties of wave forms described in Chapter II were deduced intuitively by inspection of the graphical representation shown in Figs. 1, 3, and 5. The following analytical considerations may serve to confirm the results:

It is well known that the function $y = \sin x$, when expanded into a Fourier series, gives the expression

$$y = \frac{4}{\pi}\left[\frac{1}{2} - \frac{\cos 2x}{1\cdot 3} - \frac{\cos 4x}{3\cdot 5} - \frac{\cos 6x}{5\cdot 7} - \ldots\right]. \qquad \text{(A)}$$

This series represents the function $\sin x$ in the interval $0 \leqq x \leqq \pi$ and represents the function $-\sin x$ in the interval $-\pi \leqq x \leqq 0$. For all values of x the series represents the function $|\sin x|$.

For a biphase rectified sine wave* the ordinates always retain the same sign shown positive in Fig. 1. It is therefore permissible to apply the foregoing relation for the total region $0 \leqq x \leqq K\pi$, where K may assume any value from 0 to ∞.

We may write Equation (A) in the form

$$y = \frac{4}{\pi}\left[\frac{1}{2} - \sum_{n=1}^{n=\infty} \frac{\cos 2nx}{(2n-1)(2n+1)}\right]^{\dagger} \qquad \text{(B)}$$

and apply this relation to a biphase rectified sine wave for the entire region.

The subtraction of two biphase rectified sine waves of equal amplitude shifted in phase by an angle ϕ can be expressed by the

*The term "rectified sine wave" and "rectification" in this investigation refers to processes of transformation of alternating sine waves into unidirectional sine pulses.

†Adams, Smithsonian Mathematical Tables, p. 139.

difference D of two series of the form (B) in which $(x - \phi)$ and x occur in the arguments of the cosine terms, i.e.,

$$D = \frac{4}{\pi}\left[\sum_{n=1}^{n=\infty} \frac{\cos 2n\ (x - \phi)}{(2n - 1)\ (2n + 1)} - \sum_{n=1}^{n=\infty} \frac{\cos 2nx}{(2n - 1)\ (2n + 1)}\right]. \quad (1)$$

Similarly the sum S of two phase-shifted biphase rectified sine waves can be expressed by

$$S = \frac{4}{\pi}\left[1 - \left\{\sum_{n=1}^{n=\infty} \frac{\cos 2n\ (x - \phi)}{(2n-1)\ (2n+1)} + \sum_{n=1}^{n=\infty} \frac{\cos 2nx}{(2n-1)\ (2n+1)}\right\}\right]. \quad (2)$$

Also the difference or sum M of a full sine wave and a phase-shifted biphase rectified sine wave can be expressed by

$$M = \sin x \mp \frac{4}{\pi}\left[\frac{1}{2} - \sum_{n=1}^{n=\infty} \frac{\cos 2n\ (x - \phi)}{(2n - 1)\ (2n + 1)}\right]. \quad (3)$$

These series (1), (2), and (3), respectively, were used to verify most of the relations which are enumerated in Sections 3, 4, and 5. For this purpose x in these equations was replaced by the particular values of the abscissas, characteristic for the critical points, namely, $x = 0$, $x = \phi/2$, and $x = \phi$.

The following method of verification was found to be simpler and more complete:

A full sine wave is transformed into a rectified biphase sine wave by changing the sign of its ordinates from $-$ to $+$ for the regions $(K - 1)\pi < x < K\pi$ where $K = 2, 4, 6, \ldots$.

The series (1), (2), and (3), may be replaced by the simpler relations

$$D = \text{rectified sin } x - \text{rectified sin } (x - \phi) \quad (4)$$

$$S = \text{rectified sin } x + \text{rectified sin } (x - \phi) \quad (5)$$

$$M = \text{full sin } x \quad \mp \text{rectified sin } (x - \phi). \quad (6)$$

Since rectification merely changes the sign of negative ordinates, we may replace "rectified sin x" by sin x in an interval in which sin $x > 0$, and by $-\sin x$ in an interval in which sin $x < 0$. A similar statement applies to "rectified sin $(x - \phi)$."

Another way of adding and subtracting "rectified sin x" and "recti-

fied sin $(x - \phi)$" is to regard them as sine waves, whose negative ordinates are replaced by positive ones obtained by phase-shifting by an angle $+\pi$ or $-\pi$. Accordingly the difference curve D or sum curve S of two rectified phase-shifted sine waves in an interval in which sin $(x - \phi) < 0$, may be expressed by

$$D = \sin x - \sin (x - \phi \pm \pi) \tag{4$_1$}$$

and

$$S = \sin x + \sin (x - \phi \pm \pi). \tag{5$_1$}$$

17. *Wave Forms Obtained by Subtraction.*—The subtraction of two phase-shifted biphase rectified sine waves results in a wave form D (Fig. 1) which shows two distinct regions.

In the first region, (x between 0 and ϕ), the ordinate of the phase-shifted sine pulse is reversed in sign as compared with a normal sine wave. Equation (4) must be therefore written,

$$D = \sin x - [-\sin (x - \phi)] = \sin x + \sin (x - \phi) \tag{4a}$$

$$\text{if } 0 \leqq x \leqq \phi.$$

Hence

$$dD/dx = \cos x + \cos (x - \phi).$$

The sign of the derivative is determined by observing that if $\phi \leqq \pi/2$, then $0 \leqq x \leqq \pi/2$ and $| x - \phi | \leqq \pi/2$. Hence both cos x and cos $(x - \phi)$ are positive, and $dD/dx > 0$. Also, $dD/dx > 0$ in case $\pi/2 \leqq \phi < \pi$ and $0 \leqq x \leqq \phi < \pi$. Thus it follows that in the first region the slope of the resultant D curve is positive.

In the second region (x between ϕ and π) no change in sign is produced by rectification, and Equation (4) is accordingly written

$$D = \sin x - \sin (x - \phi) \tag{4b}$$

$$\text{if } \phi \leqq x \leqq \pi$$

and

$$dD/dx = \cos x - \cos (x - \phi).$$

The sign of the derivative is determined by observing that x and $(x - \phi)$ are between 0 and π, and that $(x - \phi)$ is the smaller angle. Hence cos $x <$ cos $(x - \phi)$ and $dD/dx < 0$.

It follows that the slope of the resultant D curve is negative in the second region. The functions (4a) and (4b) are continuous, and have the same value if $(x - \phi) = 0$. Hence D is a continuous function whose derivative changes sign from $+$ to $-$ as $(x - \phi)$ increases through zero. It follows that the maximum value of D is reached when $(x - \phi) = 0$; $D = D_c =$ maximum for

$$x = x_c = \phi. \tag{7a}$$

By inserting $x = \phi$ in either of the equations (4a and 4b) an expression is obtained for the positive value of the amplitude D_c as a function of ϕ, namely,

$$D_c = \sin \phi. \tag{7b}$$

Similar exploration applied to another region of values such that $-\pi \leqq x \leqq 0$ and $-\pi \leqq (x - \phi) \leqq 0$ yields a minimum value of $D = D_a =$ minimum for

$$x = x_a = 0 \tag{8a}$$

and the negative amplitude becomes

$$D_a = \sin (-\phi) = -\sin \phi. \tag{8b}$$

In order to prove that when x is increased from 0 to π D becomes zero at two critical points, let us solve each of Equations (4a) and (4b) for x, when D is equal to zero. For the first region (4a)

$$D = \sin x + \sin (x - \phi) = 0$$

$$\sin x = -\sin (x - \phi).$$

Hence

$$D = D_b = 0 \qquad \text{for } x_b = \phi/2. \tag{9a}$$

For the second region (4b)

$$D = \sin x - \sin (x - \phi) = 0$$

$$\sin x = \sin (x - \phi).$$

Hence

$$D = D_d = 0 \qquad \text{for } x = x_d = \pi/2 + \phi/2. \tag{9b}$$

Equations (9a) and (9b) can be generalized to include angles larger than π

$$D = D_{d_{0,1,2}} = 0 \qquad \text{for } x = x_{d_{0,1,2}} = K\frac{\pi}{2} + \frac{\phi}{2} \qquad (10a)$$

$$K = 0, 1, 2, \ldots .$$

which shows that D becomes zero at angles separated by $\pi/2$. It follows that the frequency of the periodic function D is twice as great as that of the original non-rectified sine wave, or

$$T_D = T_0/2. \qquad (10b)$$

18. *Wave Forms Obtained by Addition.*—The addition of two phase-shifted biphase rectified sine waves, as expressed by Equations (2) and (5), produces a wave form S (Fig. 3) which, like the D curve, may be considered extended over two distinct regions.

In the first region (x between 0 and ϕ), the ordinates of the phase-shifted sine pulse are reversed in sign as compared with a normal sine wave. Equation (5) must therefore be written

$$S = \sin x + [-\sin (x - \phi)] = \sin x - \sin (x - \phi) \qquad (5a)$$

$$\text{if } 0 \leqq x \leqq \phi.$$

In order to explore the first region for maximum or minimum values we differentiate (5a) and determine the coordinates of the critical points:

$$\frac{dS}{dx} = \cos x - \cos (x - \phi) = 0$$

$$\cos x = \cos (x - \phi)$$

$$x = x_b = \frac{\phi}{2} \qquad (11)$$

$$\frac{d^2S}{dx^2} = -\sin x + \sin (x - \phi) < 0$$

$$\text{(maximum).}$$

Insert $x = \dfrac{\phi}{2}$ in Equation (5a)

$$S = S_b = \sin\frac{\phi}{2} - \sin\left(-\frac{\phi}{2}\right) = 2\sin\frac{\phi}{2}. \qquad (12)$$

In the second region (x between ϕ and π) no change in sign is produced by rectification. Equation (5) is accordingly written

$$S = \sin x + \sin (x - \phi) \qquad \text{if } \phi \leqq x \leqq \pi. \tag{5b}$$

In order to explore the second region for maximum or minimum values we differentiate (5b) and determine the coordinates of the critical points

$$\frac{dS}{dx} = \cos x + \cos (x - \phi) = 0$$

$$\cos x = -\cos (x - \phi)$$

$$x = x_d = \frac{\pi}{2} + \frac{\phi}{2} \tag{13}$$

$$\frac{d^2S}{dx^2} = -\sin x - \sin (x - \phi) < 0 \qquad \text{(maximum)}.$$

Insert $x = \dfrac{\pi}{2} + \dfrac{\phi}{2}$ in Equation (5b) and obtain

$$S = S_d = \sin\left(\frac{\pi}{2} + \frac{\phi}{2}\right) + \sin\left(\frac{\pi}{2} - \frac{\phi}{2}\right) = 2\cos\frac{\phi}{2}. \tag{14}$$

Where regions one and two meet

$$x = x_c = \phi \tag{15}$$

and the ordinates of S given by Equations (5a) and (5b) are equal. From Equations (5a) and (5b) it follows that the derivative of S changes sign from $-$ to $+$ as x increases through $x = \phi$. Hence the continuous function S has a minimum at $x = \phi$. By inserting $x = \phi$ in Equation (5a) we obtain

$$S = S_c = \sin \phi - \sin 0 = \sin \phi. \tag{16}$$

Another critical point

$$x = x_a = 0 \tag{17}$$

occurs where the first region ($0 \leqq x \leqq \phi$) and a third region [$(\phi - \pi) \leqq x \leqq 0$] meet. For this third region

$$S = -\sin x - \sin (x - \phi).$$

From this equation and (5a) it follows that the derivative of S changes sign from $-$ to $+$ as x increases through $x = 0$. Hence the continuous function S has a minimum value at $x = 0$. By inserting $x = 0$ in Equation (5a) we obtain

$$S_0''' = S_c = S_a = \sin \phi. \tag{18}$$

The amplitudes S_0' and S_0'' of the two pulses are obtained by subtracting S''' from their maximum values S_b and S_d, respectively, namely,

$$S_0' = S_b - S_0''' = 2 \sin \frac{\phi}{2} - \sin \phi \tag{19}$$

and

$$S_0'' = S_d - S_0''' = 2 \cos \frac{\phi}{2} - \sin \phi. \tag{20}$$

The periods of the two component pulses T_S' and T_S'' relating to the period, $T_0 = 2\pi$, of the original sine wave is determined by the difference of the abscissas which limit each of the component pulses.

For the first pulse a, b, c, this difference is

$$T_S' = x_c - x_a = \phi$$

and the ratio

$$\frac{T_S'}{T_0} = \frac{\phi}{2\pi}$$

therefore

$$T_S' = T_0 \frac{\phi}{2\pi}. \tag{21}$$

Similarly for the second pulse c, d, e,

$$T_S'' = x_e - x_c = \pi - \phi$$

$$T_S''/T_0 = (\pi - \phi)/2\pi$$

$$T_S'' = T_0 \frac{\pi - \phi}{2\pi}. \tag{22}$$

The third component is constant for constant ϕ, and its period may be regarded as infinite,

$$T_S''' = \infty. \tag{22'}$$

19. *Wave Forms Obtained by Combination of Rectified Pulses With Full Sine Waves.*—Combinations of phase-shifted rectified sine waves with full sine waves, whether obtained by addition or by subtraction, show similar wave forms and yield similar mathematical expressions. They differ from each other solely by a phase shift of 180 deg. The discussion is therefore limited to the wave form M (Fig. 5) obtained by subtraction of a rectified phase-shifted sine wave from a full sine wave.

The expressions for such a wave form are given in Section 16 by the Equations (3) and (6). The wave form may be regarded also in this case as consisting of three distinct regions.

In the first region (x between 0 and ϕ), the ordinates of the phase-shifted sine pulse are reversed in sign as compared with a normal sine wave. Equation (6) must therefore be written

$$\begin{aligned} M &= \sin x - [-\sin (x - \phi)] \\ &= \sin x + \sin (x - \phi) \qquad \text{if } 0 \leqq x \leqq \phi. \end{aligned} \tag{6a}$$

We determine the abscissas of the critical point a, at which M vanishes in this region, changing sign from $-$ to $+$. By equating Equation (6a) to zero, we obtain

$$M = \sin x + \sin (x - \phi) = 0$$

or

$$\sin x = -\sin (x - \phi)$$

$$x = x_a = \frac{\phi}{2}. \tag{23}$$

In the second region (x between ϕ and $\pi + \phi$) no change in sign is produced by rectification. Equation (6) is accordingly written

$$M = \sin x - \sin (x - \phi) \qquad \text{if } \phi \leqq x \leqq \pi + \phi. \tag{6b}$$

The next critical point

$$x = x_b = \phi \tag{24}$$

is one at which the first and the second regions meet. At this point the values of M given by Equations (6a) and (6b) are equal. From the Equations (6a) and (6b) it follows that the derivative of M changes sign from $+$ to $-$ as x increases through $x = \phi$. Hence the continuous function M has a maximum value at $x = \phi$. By inserting $x = \phi$ in either (6a) or (6b) we obtain

$$M = M_b = \sin \phi + \sin 0 = \sin \phi. \tag{25}$$

The abscissa for the third critical point c, for which $M = 0$ is determined by equating function (6b) to zero

$$M = \sin x - \sin (x - \phi) = 0$$

$$\sin x = \sin (x - \phi)$$

or

$$x = x_c = \frac{\pi}{2} + \frac{\phi}{2}. \tag{26}$$

We explore next region two for a minimum or maximum by differentiating Equation (6b).

$$\frac{dM}{dx} = \cos x - \cos (x - \phi) = 0$$

$$\cos x = \cos (x - \phi)$$

$$x = x_d = \frac{\phi}{2} + \pi \tag{27}$$

$$\frac{d^2M}{dx^2} = -\sin x + \sin (x - \phi) > 0$$

(minimum).

By inserting $x = \phi/2 + \pi$ in (6b) we obtain for the minimum point, d, the value of the ordinate

$$M_{\min}' = M_d = \sin\left(\frac{\phi}{2} + \pi\right) - \sin\left(\frac{\phi}{2} + \pi - \phi\right)$$

$$M_d = -2 \sin \frac{\phi}{2}. \tag{28}$$

Next we explore for a minimum or maximum region three, x between $(\pi + \phi)$ and $(2\pi + \phi)$. In this region the rectified phase-shifted wave B undergoes a change of sign. Equation (6) must therefore be written

$$M = \sin x + \sin (x - \phi). \tag{6c}$$

By differentiating we find

$$\begin{aligned} \frac{dM}{dx} &= \cos x + \cos (x - \phi) \\ \cos x &= -\cos (x - \phi) \\ x &= x_f = \frac{3}{2}\pi + \frac{\phi}{2} \qquad (29) \\ \frac{d^2M}{dx^2} &= -\sin x - \sin (x - \phi) > 0 \\ &\qquad \text{(minimum).} \end{aligned}$$

By inserting $x = \frac{3}{2}\pi + \frac{\phi}{2}$ in Equation (6c) we obtain for the second minimum the value of the ordinate

$$\begin{aligned} M_{\min}'' = M_f &= \sin\left(\frac{3}{2}\pi + \frac{\phi}{2}\right) + \sin\left(\frac{3}{2}\pi + \frac{\phi}{2} - \phi\right) \\ M_f &= -2 \sin\left(\frac{3}{2}\pi + \frac{\phi}{2}\right) \\ &= -2 \cos\frac{\phi}{2}. \qquad (30) \end{aligned}$$

Finally, we investigate the critical point

$$x = x_e = \pi + \phi. \tag{31}$$

At this point the second and the third ranges meet, and the values of M given by Equations (6b) and (6c) are equal. From (6b) and (6c) it follows that the derivative of M changes sign from + to − as x increases through $x = (\pi + \phi)$. Hence the continuous function

M has a maximum value at $x = (\pi + \phi)$. Inserting $x = (\pi + \phi)$ in either (6b) or (6c) we obtain

$$M = M_e = \sin(\pi + \phi) - \sin(\pi + \phi - \phi)$$
$$= \sin(\pi + \phi) = -\sin\phi. \tag{32}$$

By applying the procedure used in Section 18 for the calculation of the periods of three component pulses T_M', T_M'', and T_M''' relative to the period, $T_0 = 2\pi$, of the original sine wave, we obtain for T_M'

$$T_M' = x_c - x_a = \frac{\pi}{2} + \frac{\phi}{2} - \frac{\phi}{2} = \frac{\pi}{2}$$

$$\frac{T_M'}{T_0} = \frac{\frac{\pi}{2}}{2\pi}$$

$$T_M' = \frac{T_0}{4} \tag{33}$$

for T_M''

$$T_M'' = x_e - x_c = \pi + \phi - \frac{\pi}{2} - \frac{\phi}{2} = \frac{\pi}{2} + \frac{\phi}{2}$$

$$\frac{T_M''}{T_0} = \frac{\frac{\pi + \phi}{2}}{2\pi}$$

$$T_M'' = T_0 \frac{\pi + \phi}{4\pi} \tag{34}$$

for T_M'''

$$T_M''' = x_{a_1} - x_e = 2\pi + \frac{\phi}{2} - \pi - \phi = \pi - \frac{\phi}{2}$$

$$\frac{T_M'''}{T_0} = \frac{\pi - \frac{\phi}{2}}{2\pi}$$

$$T_M''' = T_0 \frac{2\pi - \phi}{4\pi}. \tag{35}$$

20. *Relations Between Wave Forms.*—The question of the character of the component parts of the D, S, and M curves can now be answered.

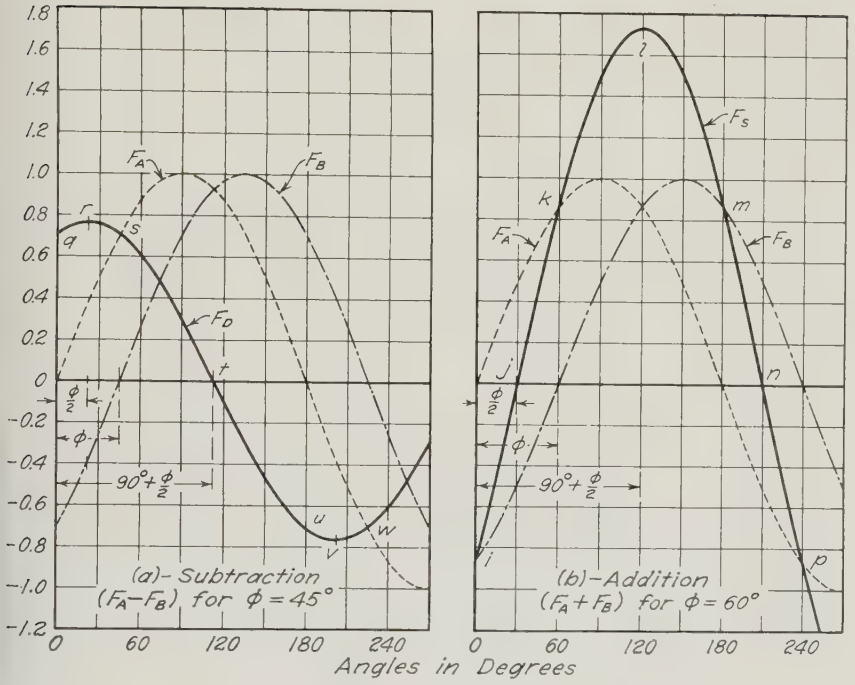


Fig. 15. Subtraction and Addition of Two Full Sine Waves

In Table 4 are listed the Equations (4a) to (6c) for the component parts of the D, S, and M curves and the respective ranges of abscissae for which they are valid. For better identification each component part is marked by letters referring to the graphical presentation in Figs. 1, 3, and 5. Comparison of the equations and the ranges for the four component parts of the M curve with those for the D and S curves leads to the following conclusions:

Wave form M_{hab}, Equation (6a), is identical with curve D_{abc}, Equation (4a) (in phase)

Wave form M_{beg}, Equation (6b), is identical with curve D_{cda}, Equation (4b) (in phase)

Wave form M_{gde}, Equation (6b), is identical with curve S_{abc}, Equation (5a) (phase shifted 180 deg.)

Wave form M_{efh}, Equation (6c), is identical with curve S_{cde}. Equation (5b) (phase shifted 180 deg.)

TABLE 4
COMPARISON OF D, S, AND M CURVES

Wave Form	Points Marking Comp. Parts	Fig.	Range of Abscissa	Equations for Component Parts	Equation	Identity Marks	Wave Form	Points Marking Comp. Parts	Fig.
D	a, b, c	1	$0 \leqq x \leqq \phi$	$\sin x + \sin(x - \phi)$	(4a)	○	FS	i, j, k	15b
D	c, d, a_1	1	$\phi \leqq x \leqq \pi$	$\sin x - \sin(x - \phi)$	(4b)	□	FD	s, t, u	15a
S	a, b, c	3	$0 \leqq x \leqq \phi$	$\sin x - \sin(x - \phi)$	(5a)	△	FD	g, r, s	15a
S	c, d, e	3	$\phi \leqq x \leqq \pi$	$\sin x + \sin(x - \phi)$	(5b)	×	FS	k, l, m	15b
M	h, a, b	5	$0 \leqq x \leqq \phi$	$\sin x + \sin(x - \phi)$	(6a)	○	FS	i, j, k	15b
M	b, c, g	5	$\phi \leqq x \leqq \pi$	$\sin x - \sin(x - \phi)$	(6b)	□	FD	s, t, u	15a
M	g, d, e	5	$\pi \leqq x \leqq (\pi + \phi)$	$\sin x - \sin(x - \phi)$	(6b)	△	FD	u, v, w	15a
M	e, f, h_1	5	$(\pi + \phi) \leqq x \leqq 2\pi$	$\sin x + \sin(x - \phi)$	(6c)	×	FS	m, n, p	15b

Further proof of the identity is derived by comparing the expressions for the amplitudes M_b, M_d, and M_f in Table 3 with the corresponding expressions for D_c in Table 1, and with the expressions for S_b and S_d in Table 2. Thus the equality of the respective absolute values of the amplitudes are obtained, namely

$$M_b = D_c;\ M_d = S_b;\ \text{and}\ M_f = S_d.$$

By the way of synthesis all the component parts enumerated on the left side of Table 4 may be combined to form a difference curve FD and a sum curve FS of two full sine waves (F_A and F_B respectively) shifted in phase by an angle ϕ. The corresponding expressions for these wave forms are

$$FD = \sin x - \sin\,(x - \phi)$$

and

$$FS = \sin x + \sin\,(x - \phi).$$

They are shown graphically in Figs. 15a and 15b for phase angles $\phi = 45$ deg. and $\phi = 60$ deg., respectively. The points where the component parts join each other sequently are marked with the letters i to w. In the last two columns of Table 4 the component parts are listed opposite the corresponding wave forms D, S, and M, thus indicating their common origin.

VI. Applications

21. *Review of Applications.*—The main field of applications of the wave forms described in the preceding chapters relates to the measurement of the phase angle between two sine waves throughout the lower, the audio, and the radio frequency spectrums. Other applications include the use of these new wave forms in square wave oscillators, harmonic oscillators, frequency multipliers, and in sweep circuits for use in oscillographs, television, and electronic time-control devices.

The various applications and the forms in which they are carried out, will be described elsewhere. Only the principles and some of the particular methods of phase measurements are included in the following discussion.

22. *Oscillographic Method of Phase Angle Measurement Using Difference Curve D.*— A simple method of phase angle measurements using the difference curve D consists in observing and evaluating the coordinates of the critical points of the wave forms traced on the screen of the oscillograph, making use of the relations (8b) and (7b) in Table 1 (see also Fig. 1). It can be seen that the amplitudes of the difference curve are proportional to the sine of the phase angle ϕ. By measuring on the oscillograph screen the length $\overline{cc_1}$ where c_1 is the intersection of a perpendicular dropped from the maximum point C to the X axis and measuring the amplitude of the rectified sine wave, $\overline{A}$, the phase angle in radians is given by the relation

$$\frac{\overline{cc_1}}{\overline{A}} = \sin\phi \qquad \text{or} \qquad \phi = \sin^{-1}\frac{\overline{cc_1}}{\overline{A}}.$$

A more direct and accurate method makes use of the relation (9a) in Table 1 (see Fig. 1*e*). A perpendicular line is projected from the maximum point c to the intersection c_1 on the X axis. By measuring the lengths $\overline{bc_1}$ and $\overline{bd}$, the phase angle in radians is given by the relation

$$\phi = \pi\frac{\overline{bc_1}}{\overline{bd}}.$$

Since the points b, c_1, and d are well defined, these distances can be fairly accurately determined and the angles computed from the fore-

going relation. The limitations of accuracy are set by the sharpness of the focus of the electronic beam and the linearity of the time axis of the sweep circuit.

23. *Oscillographic Method of Phase Angle Measurement Using Sum Curve S.*—The oscillographic method becomes even more simplified if the sum curve S (Fig. 3) is used in place of the difference curve D. Making use of the relations (17) and (15) in Table 2 it follows that the length $\overline{ac}$ is related to the phase angle as the length $\overline{ae}$ is related to π. Therefore

$$\phi = \pi \frac{\overline{ac}}{\overline{ae}}.$$

The points a, c, and e are very well defined, and the respective distances between them can be easily measured. Therefore, the phase angle ϕ can be determined accurately.

Any of the other relations valid for the critical points, as enumerated in Table 2, may be used. For example, from relation (18) it follows that since the d.c. component S_0''' is proportional to sin ϕ, the phase angle can be determined by obtaining the distance of any of points a, c, or e to the X axis, and using the relation

$$\phi = \sin^{-1} \frac{S_0'''}{A_0}.$$

24. *Oscillographic Method of Phase Angle Measurement Using Composite Curve M.*—In this case the critical points b and e are the most practical points for accurate determination of the phase angle. The relation (25) in Table 3 gives the maximum ordinate $\overline{bb_1}$, which is proportional to the sin ϕ. This property is similar to that mentioned in Section 21 for the maximum ordinate of the difference curve D. Therefore, knowing the lengths $\overline{A}$ and $\overline{bb_1}$, the phase angle is computed from

$$\phi = \sin^{-1} \frac{\overline{bb_1}}{\overline{A}}.$$

Another method consists in measuring the relative lengths $\overline{ab_1}$ and $\overline{ac}$ and using the relation

$$\phi = \pi \frac{\overline{ab_1}}{\overline{ac}}.$$

The critical points e and g may also be applied for phase angles between 30 and 90 deg. In this case a line parallel to the X axis is drawn between the points e and h. The distance $\overline{ge}$ is related to the phase angle ϕ as the length $\overline{hg}$ is related to π. Therefore

$$\phi = \pi \frac{\overline{ge}}{\overline{hg}}.$$

25. *Phase Angle Measurement by Means of Direct-Reading Instruments.*—A direct-reading method has been developed in order to avoid the necessity for measuring lengths on the oscillographic screen and of making computations to obtain the phase angles. In this connection use is made of any of the simple relations enumerated in Tables 1, 2, and 3, so that known commercial indicating meters can be adapted to the mixing circuits shown in Figs. 7, 9, and 10. Indicating instruments such as thermionic voltmeters for measuring crest voltages, or a.c. instruments for measuring the integrated values of the D curve, or galvanometers for measuring the d.c. component of the S curve are all equally applicable.

In all cases the condition must be fulfilled that the amplitudes, A and B, of the rectified pulses be of equal magnitudes. This makes it necessary that current-limiting devices be installed for keeping these amplitudes constant and equal to each other.

A detailed account of these methods and devices will be published elsewhere.

VII. Résumé of Investigation

26. *Summary.*—The researches described in this bulletin were partly theoretical and partly experimental. The task of the theoretical investigation was to find and to formulate composite wave forms made up of component sine waves, having sharply marked critical points, and also to derive relations which could be used to determine the phase angles of the component waves.

The experimental part consisted of developing methods of producing such composite wave forms and utilizing them for the determination of phase angles of any two given phase-shifted sine waves of like frequency. Further, the investigation necessitated the development of accurate methods of producing arbitrarily adjustable phase shifts.

These researches finally led to a number of applications which may prove useful in the field of electronics.

27. *Results.*—The results of the investigation may be stated as follows:

(1) By combining two phase-shifted rectified sine waves, new wave forms have been obtained. It was found that the critical points of these wave forms depend upon the phase angle between the component rectified sine waves.

(2) Three types of such wave forms have been investigated:

(a) those obtained by the subtraction of two rectified phase-shifted sine waves,

(b) those obtained by the addition of two rectified phase-shifted sine waves,

(c) those obtained by a rectified phase-shifted sine wave added to or subtracted from a full sine wave.

(3) Expressions for the coordinates of the critical points have been formulated for all three types of wave forms.

(4) The variation of the wave shapes as a function of the phase angle has been investigated graphically and analytically.

(5) Various methods have been experimentally studied for the production of phase-shifted rectified sine waves, each adaptable for a definite frequency range.

(6) For the low frequency range, from 30 to 250 cycles per second, the method of vector addition in quadrature relation has been adopted for producing phase-shifted sine waves.

(7) A method based on sound radiation has been developed for phase-shifting in the audio frequency range from 250 to 20 000 cycles per second.

(8) For the radio frequency range between the limits of 20 000 and 100 000 cycles per second the method of phase-shifting by varying the constants of the circuits coupled to the oscillator has been applied.

(9) Circuits for the observation of the new wave forms have been devised and applied.

(10) The properties of the wave shapes have been experimentally verified by means of oscillograms.

(11) By utilizing the properties of the critical points, oscillographic methods of determining phase differences have been developed for all three types of wave forms.

(12) Direct-reading methods of measuring phase differences have been devised.

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